MAT 160 Spring 2010

Puzzles and paradoxes with probability - homework.

1. The Two Envelope Paradox (adapted from Wikipedia).

   The setup: The player is given two indistinguishable envelopes, each of which contains a positive sum of money. One envelope contains twice as much as the other. The player may select one envelope and keep whatever amount it contains, but upon selection, is offered the possibility to take the other envelope instead.

   The switching argument: Denote by A the amount in the selected envelope. The probability that A is the smaller amount is $\frac{1}{2}$, and that it's the larger is also $\frac{1}{2}$. The other envelope may contain either $2A$ or $\frac{A}{2}$. If A is the smaller amount, the other envelope contains $2A$. If A is the larger amount, the other envelope contains $\frac{A}{2}$. Thus, the other envelope contains $2A$ with probability $\frac{1}{2}$ and $\frac{A}{2}$ with probability $\frac{1}{2}$. So the expected value of the money in the other envelope is:

   $$\frac{1}{2} \cdot 2A + \frac{1}{2} \cdot \frac{A}{2} = \frac{5}{4}A.$$

   This is greater than A, so swapping is favored.

   BUT: after the switch, reason in exactly the same manner as above, but denote the second envelope's contents as B. It follows that the most rational thing to do is to swap back again. Explain this apparent paradox.

2. Bertrand's box paradox (adapted from Wikipedia)

   This is a classic paradox of elementary probability theory. It was first posed by Joseph Bertrand in his *Calcul des Probabilités*, published in 1889.

   There are three boxes: a box containing two gold coins, a box with two silver coins, and a box with one of each. You choose a box at random and take out a coin at random: it is gold. What is the probability that the remaining coin is gold? It seems to be $\frac{1}{2}$; in fact, the probability is actually $\frac{2}{3}$. Explain.

[In a 1950 article, Warren Weaver introduced a simple way to conduct the experiment on people: the boxes are replaced by cards, and gold and silver coins are replaced by red and black markings, one marking placed on each of the two faces of each card. In what Martin Gardner has called the three-card swindle, a card is drawn from a hat, and if a red mark is shown, the dealer bets the victim even money that the other side is also red. The victim is convinced that the bet is fair, but the dealer makes money in the long run by winning $\frac{2}{3}$ of the time.]

3. An example of the Borel−Kolmogorov paradox. (adapted from Wikipedia).

   On the surface of the Earth, if you know your latitude is zero, what is the probability that your longitude is within 30 degrees of zero (i.e. between 30°E and 30°W)? If you know your longitude is zero, what is the probability that the latitude is within 30 degrees of zero (i.e. between 30°N and 30°S)? Explain why these are not the same.

4. Littlewood's law: "With a sample size large enough, any outrageous thing is likely to happen." (Adapted from Wikipedia).

   J.E. Littlewood defines a *miracle* as an exceptional event of special significance occurring at a frequency of one in a million. He assumes that during the hours in which a human is awake and alert, a human will experience one event per second, which may be either exceptional or unexceptional (for instance, seeing the computer screen, the keyboard, the mouse, the article, etc.). Additionally, Littlewood supposes that a human is alert for about eight hours per day.

   As a result, a human will, in 35 days, have experienced, under these suppositions, 1,008,000 events. Accepting this definition of a miracle, one can be expected to observe one miraculous occurrence within the passing of every 35 consecutive days and therefore, according to this reasoning, seemingly miraculous events are actually commonplace.

   Give an application of this observation to every-day life.