MAT 160 Spring 2010

Homework 2. Solutions.

1. (from discussion of "Binary 101" card trick).

In "down under" dealing the top card is dealt and the second card put at the bottom of the deck, and so forth.

Prove that if n cards are dealt "down under," the last card dealt is the card that was originally $\varphi(n)$ from the top, where $\varphi(n)$ is equal to twice the difference between n and the greatest power of 2 which is $< n$.

We checked this for $n = 2^k$: In that case $\varphi(n) = 2(2^k - 2^{k-1}) = 2^k$; as you deal you first deal all the odd-numbered cards, then those numbered of the form $2\times$ odd, then those numbered of the form $4\times$ odd; finally the only card left is the one numbered $2^k \times 1$, so the initial bottom card is dealt last.

- It is helpful to 1. Express the original number of cards as a binary number and 2. Think of "down-under" dealing as producing a series of shorter and shorter decks, with the last one a 1-card deck = the last card. A new deck starts when as many deal one/save one pairs have been dealt with the previous deck. This means that if the previous deck has an odd number of cards, then the bottom card is dealt and the previous number-2 card goes to the bottom of the deck. If the previous deck has $2n$ cards, the new deck has n cards, and the same bottom card as the previous deck; if the previous deck has $2n + 1$ cards, the new deck has n cards, but its bottom card is the previous deck's second card. Finally 3. notice that if the cards in deck 0 (the original deck) are numbered 1 to N, then in deck 1 the cards are numbered 2 units apart (except perhaps at the end); in deck 2 they are 4 units apart, etc.

  (2.) implies that the number of cards in deck $n + 1$, in binary, comes from dropping the last digit from the number of cards in deck n. (2.) and (3.) imply that when that last digit was a 1, the new bottom card has original rank 2^n larger than the previous one. So the original rank of the last bottom card is the sum of 2^{k+1} for each k such that the original deck cardinality, written in binary, has a 1 in the kth position, counting from the right starting with 0. The leftmost 1 does not count because the deal stops there. This is equivalent to the statement in the problem.

For example if the original deck has 100101 (37) cards, the successive decks will have 10010, 1001, 100, 10, 1 cards. The last card in deck 1 is the old #2; that is also the last card in deck 2. The last card in deck 3 is $2^3 = 8$ places behind the last card in deck two, so it is the original #10 (ten). That remains the last card until the end. Notice that in binary, the original rank of the last card is 2 (10) plus 8 (1000) = 1010. This number can be obtained directly from 37 (100101) by chopping off the initial 1 and tacking a 0 at the end, and this works in general.

2. A microbe either splits into two perfect copies of itself or else disintegrates. If the probability of splitting is p, what is the probability that one microbe will produce an everlasting colony?

   - The key is to realize that a cell will produce an everlasting colony if it splits and one of its daughters produces an everlasting colony, i.e. if it splits and neither of its daughters does not p.a.e.c. So if x is the probability that a cell p.a.e.c., then $x = p[1 - (1-x)^2]$. This gives $x = p[2x - x^2]$ or $px^2 = (2p-1)x$. The solutions are $x = 0$ and $x = \frac{2p-1}{p}$. So for example if $p = \frac{3}{4}$ then $x = \frac{2}{3}$.

3. A sequence $y_1, y_2, y_3, ...$ of real numbers is *monotone* if either $y_1 \leq y_2 \leq y_3 \leq ...$ or $y_1 \geq y_2 \geq y_3 \geq ....$ Prove that every sequence of real numbers has a monotone subsequence.

   - Call an entry y_k in the sequence a *peak* if it is $\geq$ all the entries coming after it in the sequence. If there are infinitely many peaks, then the peaks form a monotone decreasing subsequence. If there are only finitely many peaks, say ending with y_N, then beyond y_{N+1} there is an entry $y_{N+k} > y_{N+1}$ (otherwise y_{N+1} would be a peak). And further on there is an entry $> y_{N+k}$, etc. These elements form a monotone increasing subsequence.

4. Devise an experiment which uses only tosses of a fair coin, but which has success probability $\frac{1}{3}$. Do the same for any success probability p,

$0 \leq p \leq 1$.

- Write $p = 2^{-a} + 2^{-b} +$ etc. (i.e. express p as a possibly infinite binary decimal). Then define success in the experiment to mean that the first head comes on the ath toss or the bth toss etc. In the case $p = \frac{1}{3} = 0.010101...$, success means that the first head occurs on an even-numbered throw.

5. If a set of positive integers has sum n, what is the biggest its product can be?

    - Split n into 3s, with one or two 2s as necessary. So $8 = 3+3+2$, with product 18, is the best you can do. To see why this works, suppose one of the terms was $m \geq 4$. Then splitting m as $2 + (m-2)$ gives product $2m - 4 \geq m$ so we can make m smaller with no loss. In the sum there can't be more than two 2s, because $2 + 2 + 2 = 3 + 3$ and $8 < 9$.