Impossible configurations for geodesics on negatively-curved surfaces

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Abstract Hass and Scott's example of a 4-valent graph on the 3-punctured sphere that cannot be realized by geodesics in any metric of negative curvature is generalized to impossible configurations filling surfaces of genus n with k punctures for any n and k.

1 Introduction

By a configuration on a surface S we mean a 4-valent, connected graph embedded in S. Going straight (neither right nor left) at each intersection decomposes any configuration canonically into a collection of closed curves (the *tracks* of the configuration) intersecting themselves and others transversally.

A basic question is whether or not there is a hyperbolic metric on S such that the configuration is isotopic to a collection of closed geodesics intersecting transversally. We will say in this case that the configuration can be *realized* by geodesics. It is an old but remarkable fact that the simple configuration shown in Figure 1 cannot be realized by a geodesic in any metric of negative curvature. Joel Hass and Peter Scott discovered this phenomenon in 1999 [1]. As they remark, their proof of non-realizability can be by replaced by an argument, due to Ian Agol, using the Gauss-Bonnet Theorem.

In this paper Agol's argument is generalized to produce an infinite family of non-realizable configurations, the *polygonal impossible configurations*, including a one-track, non-realizable configuration filling the surface of genus n with k punctures for every $n \geq 2$ and $k \geq 0$. In some sense polygonal impossible configurations are all of the non-realizable examples that can be constructed using our general form of the argument.

1.1 Preliminaries

We will construct configurations with the following properties.

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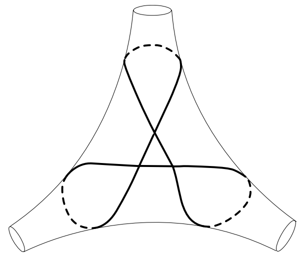

Figure 1: The Hass-Scott example.

- The first property, required by negative curvature, is that none of the curves represents a power (> 1) in the free homotopy group $\pi_1 S$, nor do two distinct curves represent powers (≥ 1) of the same element of $\pi_1 S$. Since in negative curvature each free homotopy class contains a unique geodesic, a power curve collapses to multiple tracings of a single geodesic, and two homotopic curves collapse to the same geodesic; in either case the initial configuration is destroyed.
- The second is that the configuration *fills* the surface in the sense that every complementary region is topologically either a disc or a once-punctured disc. (For a graph embedded in a smooth surface [3], this corresponds to the property of being *cellular*; when the configuration is one-track, it produces a *filling curve* [2]). Without this filling property one easily gets examples of impossible configurations with any genus or number (≥ 3) of deleted points directly from the Hass-Scott example by adding punctures and handles.

2 Polygonal impossible configurations

Definition 2.1. A *polygonal impossible configuration* $\mathcal{P}$ is any orientable, connected 2-dimensional cellular complex constructed as follows:

(1) Choose a number $N \geq 3$, which will be the number of vertices in the configuration.

(2) Choose a number p, with $N/4 \leq p \leq N/3$, and p polygons $A_1, \dots, A_p$ which together have N corners (This is possible since $p \leq N/3$). Furthermore, at least one A_i must be a triangle: see the remark below.

(3) Choose $q = N - 2p$ even-sided polygons $B_1, \dots, B_q$ which together have $2N$ corners. (This is possible since $p \geq N/4$ implies $4q = 4N - 8p \leq 2N$).

(4) Identify an edge of one of the A_i with every other edge of each B_j, preserving orientations. Avoid forming a ring of squares: such a ring would lead to two parallel tracks.

■

Remark 2.2. At least one of the A_i must be a triangle. In fact, suppose first all the A_i are squares; then $N = 4p, q = N - 2p = 2p$, and $2N/q = 4$, so all the B_i must also be squares; and then the configuration $\mathcal{P}$ constructed by the algorithm will be made up of one or more sets of parallel curves, so cannot be non-power. On the other hand if all the A_i have ≥ 4 sides, and at least one has strictly more, then $N > 4p$ contradicting $p \geq N/4$. ■

Theorem 2.3. *A polygonal impossible configuration cannot cannot be embedded in a surface of negative curvature so that the set of curves defined by its 1-skeleton is a set of geodesics.*

Proof. Set n_i to be the number of vertices of A_i, and $\alpha_{i,j}$, $j = 1, \dots, n_i$ to be the interior angle at the jth vertex of A_i.

Likewise set m_i to be the number of vertices of B_i, and $\beta_{i,j}$, $j = 1, \dots, m_i$ to be the interior angle at the jth vertex of B_i.

Assume all the edges are geodesic arcs extending smoothly from polygon to polygon, so that each of the $\alpha_{i,j}$ is complementary to exactly two of the $\beta_{i,j}$.

The Gauss-Bonnet theorem [7] gives

$$\begin{array}{rcl} \alpha_{1,1} + \alpha_{1,2} + \cdots \alpha_{1,n_1} & < & (n_1 - 2)\pi \\ & \cdots & \\ \alpha_{p,1} + \alpha_{p,2} + \cdots \alpha_{p,n_p} & < & (n_p - 2)\pi. \end{array}$$

Adding these equations,

$$\sum_{i=1}^{p} \sum_{j=1}^{n_i} \alpha_{i,j} < (N - 2p)\pi. \qquad (*)$$

Similarly, the sum of all the βs is strictly less than $(2N - 2q)\pi$. On the other hand each β is $\pi - \alpha$ for some α, with each α occurring exactly twice.

So

$$(2N - 2q)\pi > \sum_{i=1}^{q} \sum_{j=1}^{m_i} \beta_{i,j} = 2 \sum_{i=1}^{p} \sum_{j=1}^{n_i} (\pi - \alpha_{i,j}) = 2N\pi - 2 \sum_{i=1}^{p} \sum_{j=1}^{n_i} \alpha_{i,j}$$

i.e. $\sum_{i=1}^{p} \sum_{j=1}^{n_i} \alpha_{i,j} > q\pi$. Since by the construction $q = N - 2p$, this inequality contradicts $(*)$. ■

3 The genus of a polygonal impossible configuration; minimal configurations

Consider a polygonal configuration $\mathcal{P}$ created from N, $A_1, \dots, A_p$ and $B_1, \dots B_q$ as above. Topologically, $\mathcal{P}$ is an orientable surface with boundary: the N unused edges of the Bs are grouped by the identifications in step 4 above into a certain number

$\gamma_1, \ldots, \gamma_r$ of closed curves, which together form the booundary $\partial\mathcal{P}$. Adding a disc along each γ_i creates a closed, orientable surface $S_{\mathcal{P}}$, with Euler characteristic $\chi = N - 2N + (p+q+r) = -p+r$, and genus

$$g_{\mathcal{P}} = \frac{1}{2}(2-\chi) = \frac{1}{2}(2-r+p) \qquad (*).$$

We can take $g_{\mathcal{P}}$ as the genus of $\mathcal{P}$; this matches the usual definition of the genus of a graph as the genus of the simplest surface on which it can be embedded so that its complement is topologically a set of discs. To make $S_{\mathcal{P}}$ into a hyperbolic surface on which $\mathcal{P}$ gives an impossible, filling configuration it suffices to puncture each monogon or bigon of $S_{\mathcal{P}}$ and, in case $g_{\mathcal{P}} = 0$ or 1, to add punctures to one or more of the discs.

We will call a polygonal configuration on a smooth surface *minimal* if it fills the surface, and has the smallest possible number of vertices. Since $r \geq 1$, $(*)$ implies that p, the number of A-polygons, must satisfy $p \geq 2g-1$. And since each A-polygon is at least a triangle, the minimum number of vertices for a polygonal impossible configuration filling the smooth surface of genus $g \geq 2$ is $6g-3$.

4 Examples

The smallest possible N is $N = 3$. Here p and q must equal 1, with A_1 a triangle and B_1 a hexagon. There are two ways to make the identification, with different results, as shown in Figure 2.

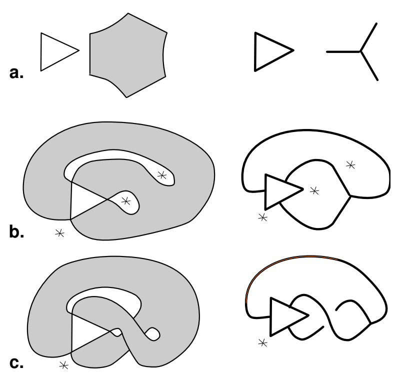

Figure 2: $N = 3$. a. one triangle, one hexagon. b. (Asterisks represent punctures). Here the identification gives $r = 3$ and $g_{\mathcal{P}} = 0$, yielding a (1-track) impossible configuration on the 3-punctured sphere (the configuration exhibited by Hass and Scott). c. Otherwise the identification gives $r = 1$ and $g_{\mathcal{P}} = 1$; hence a graph on the torus which becomes an impossible (3-track) configuration on the punctured torus. This figure illustrates the shorthand diagrams we will be using to describe polygonal impossible configurations. Each $(2n) - B$-polygon is represented by a tree with n ends. The more "realistic" image is straightforward to reconstruct from the diagram.

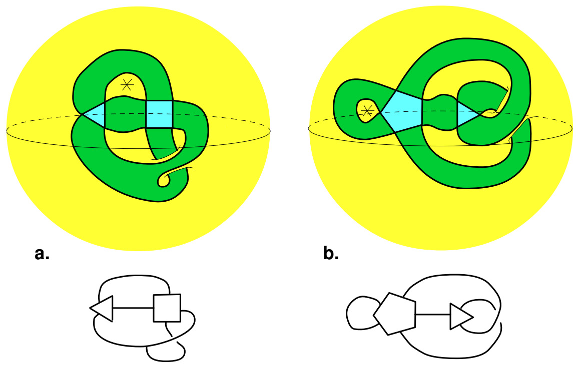

Figure 3: With the same conventions, a 1-track (a.) and a 2-track (b.) polygonal impossible configuration on the punctured torus.

5 Unicursal, filling configurations

Definition 5.1. A configuration is *unicursal* if it has exactly one track, i.e., as described in the introduction, if it can be traversed by a single curve. ∎

5.1 Smooth surfaces of genus ≥ 2

Proposition 5.2. *There exists a minimal unicursal impossible polygonal configuration filling the smooth surface of genus n.*

Proof. We begin with genus 2.

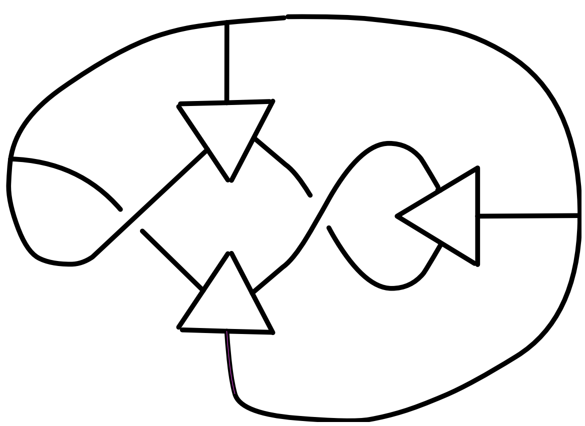

Figure 4: A 1-track impossible configuration on the genus-2 surface with $N = 9, p = 3, q = 3$ (decagon and 2 squares), $r = 1$, shown with its diagram.

This construction can be extended to give an impossible configuration $\mathcal{P}_n$ filling the surface of genus n, for each $n \geq 2$.

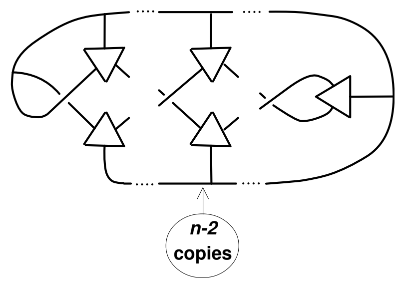

Figure 5: The configuration $\mathcal{P}_n$ has $p = 2n-1$ triangles matched with $2n-2$ squares and a $(4n+2)$-gon. It has $r = 1$ and therefore genus $\frac{1}{2}(2+p-r) = n$.

Assertion: The configuration $\mathcal{P}_n$ has a single track if $n \equiv 0$ or $2 \bmod 3$, and 3 tracks if $n \equiv 1 \bmod 3$.

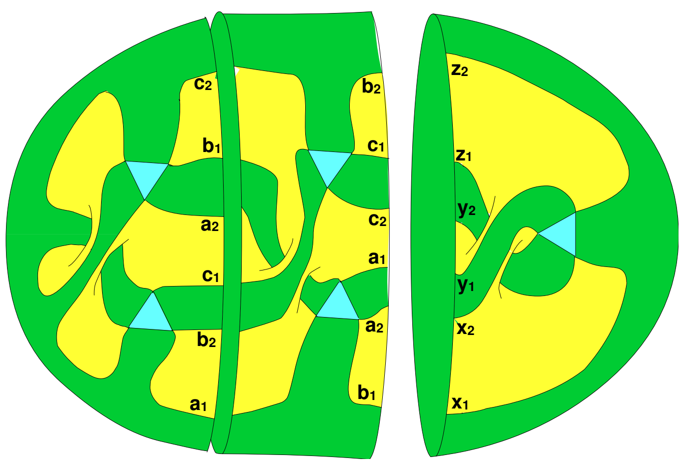

Figure 6: The two ends of $\mathcal{P}_n$, with track segments labeled.

In fact, at each stage in the assembly (from left to right) there are 3 tracks of the curve, $\mathbf{a_1a_2}, \mathbf{b_1b_2}$ and $\mathbf{c_1c_2}$. Adding the next stage permutes the six free ends by the cycle $(\mathbf{a_1b_1c_1})(\mathbf{a_2c_2b_2})$. After 0 iterations ($n = 2$), the ends $\mathbf{a_1}, \mathbf{b_2}, \mathbf{c_1}, \mathbf{a_2}, \mathbf{b_1}, \mathbf{c_2}$ will be joined respectively to the free ends on the right $\mathbf{x_1}, \mathbf{x_2}, \mathbf{y_1}, \mathbf{y_2}, \mathbf{z_1}, \mathbf{z_2}$ resulting in a single loop:

$$\mathbf{x_1a_1a_2y_2y_1c_1c_2z_2z_1b_1b_2x_2}.$$

After 1 iteration ($n = 3$), there still is a single loop:

$$\mathbf{x_1b_1b_2z_2z_1c_1c_2y_2y_1a_1a_2x_2}.$$

But after 2 iterations ($n = 4$) the identifications give three loops: $\mathbf{x_1c_1c_2x_2}$, $\mathbf{y_1b_1b_2y_2}$, $\mathbf{z_1a_1a_2z_2}$. Since the permutation has order 3, the sequence repeats.

The case of genus $3k+4$, $k \geq 0$. In this case a different construction produces a family of minimal, unicursal, filling configurations. It is shown in Figure 7. ■

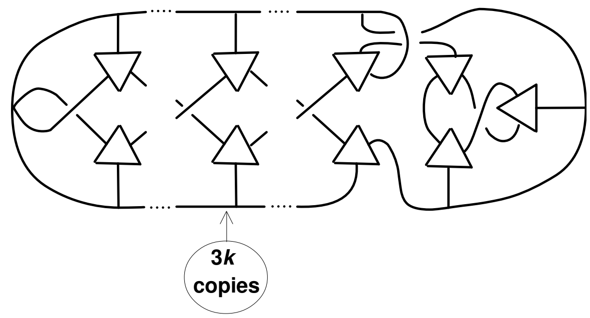

Figure 7: The configuration $\mathcal{Q}_{3k+4}$, $k \geq 0$ has $6k+7$ A-polygons, all triangles, so $N = 18k+21$ and $q = 6k+7$. The B-polygons are $6k+6$ squares and one $12k+18$-gon. One can check this configuration is unicursal and that $r = 1$, so the genus is $\frac{1}{2}(2+p-r) = 3k+4$ and the configuration is filling.

5.2 Surfaces with punctures

Theorem 5.3. *The number of tracks in a polygonal configuration is congruent mod 2 to the number of vertices. In particular, a configuration can be unicursal only if the number of vertices is odd.*

This theorem is proved in the Appendix.

For configurations filling surfaces with two or more punctures, this theorem produces a conflict between being unicursal and being minimal, as defined above. Suppose that $\mathcal{P}$ (N, p, q, r) is a minimal, unicursal configuration on the smooth surface $\Sigma_{n,0}$ (we use the notation $\Sigma_{n,k}$ for the surface of genus n with k punctures). By minimality and $(*)$, $p = 2n-1$. Since $r = 1$ the surface can be given one puncture in that complementary region; the result will be a minimal, unicursal configuration filling $\Sigma_{n,1}$. No problem so far. Working on $\Sigma_{n,2}$ we will need $r = 2$ and, by $(*)$, $p = 2n$. Minimality would require all p of the A-polygons to be triangles, but then $N = 3p$ would be even and, by Theorem 1, the configuration cannot be unicursal. Both possibilities can be realized separately, by adding to the configutation $\mathcal{P}$ one or the other of the sub-configurations shown in Fig. 8.

This discussion can be summarized as follows:

Proposition 5.4. *There exist a minimal polygonal impossible configuration and a unicursal polygonal impossible configuration filling the surface of genus n with k punctures, $n \geq 2, k \geq 0$. These are generally different.*

5.3 Genus 0 and 1

For genus 0 the unicursal and minimal (Hass-Scott) example on the 3-punctured sphere can be extended by splicing in copies of Fig. 8 a. (resp. b.) to give a minimal (resp. unicursal) polygonal impossible configuration on the sphere with k punctures ($k \geq 3$).

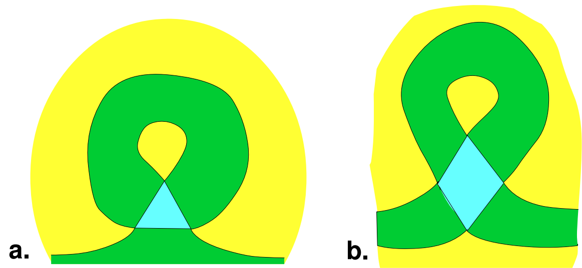

Figure 8: These two sub-configuration can be added to a configuration $\mathcal{P}$ filling $\Sigma_{n,k}$. Sub-configuration a. can be grafted along a free edge of any B-polygon; b. can be inserted at any of the interfaces between a B-polygon and an A. The result is a configuration on the same surface, but with r increased by 1. Puncturing the new complementary cell will exhibit the new configuration as a polygonal impossible configuration filling $\Sigma_{n,k+1}$. If $\mathcal{P}$ was minimal, using a. will also give a minimal configuration (this follows from $(*)$); but it will clearly not be unicursal. If $\mathcal{P}$ was unicursal, using b. will also give a unicursal configuration; but it will not be minimal.

For genus 1 the minimal example of Fig. 2 b. can be extended by splicing in copies of Fig. 8 a. to give a minimal polygonal impossible configuration on the torus with k punctures ($k \geq 1$); similarly for unicursals, using Fig. 3 a. and Fig. 8 b.

6 Appendix: Unicursal configurations

The examples in Figs. 2, 3 and 4 suggest the following statement.

Theorem 6.1. *The number of tracks of a polygonal impossible configuration is congruent mod 2 to the number of vertices. In particular, such a configuration can only be unicursal if the number of vertices is odd.*

Preliminaries for the proof.

(1) Taking the planar polygons A_i and B_i as in the construction of the configuration, give each one the standard (counterclockwise) orientation. Then give the segments of the configuration their inherited orientation, except segments shared by an A and a B keep their A-orientation. With this convention, the track-segments at each intersection are coherently oriented, and give a well-defined orientation on each track (Figure 9).

(2) Project the configuration into the plane, and consider it as a collection of oriented immersed curves. The configuration depends only on the nature of the polygons A_i and B_i and the way they are connected. In particular, its projection can be displayed so that the A_i and B_i appear as in Figure 10.

Proof of Theorem 6.1

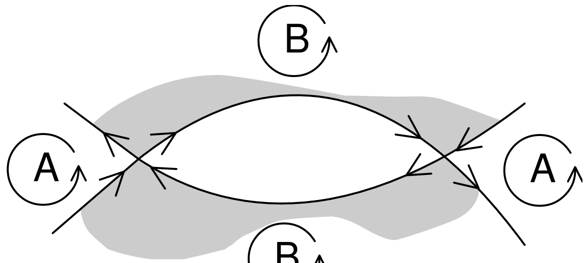

Figure 9: The A and B orientations are adjusted to give a well-defined orientation on each track.

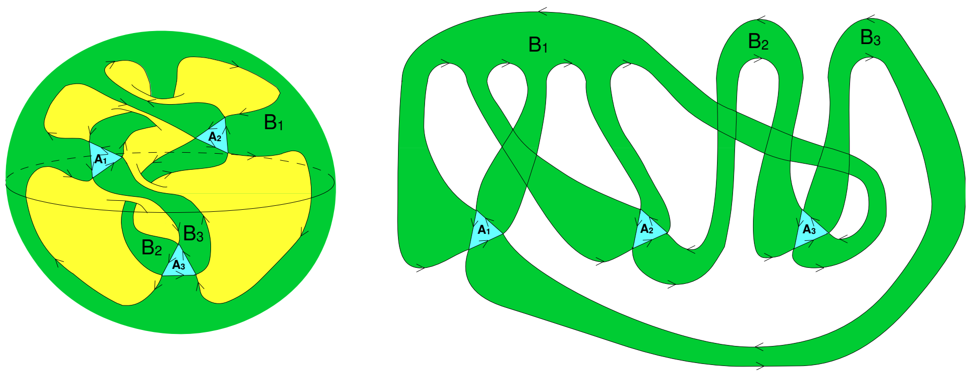

Figure 10: The projection of a polygonal impossible configuration can be displayed with all the B polygons at the top, all the A polygons at the bottom, and so that the only horizontal tangents appear at the top and at the bottom. This figure shows the configuration from Figure 4, oriented as above, with its display in this form.

(1) We show that the sum of the rotation numbers of the projected complex of curves is even. This is the sum of the degrees of the Gauss maps, which take a parameter value to the unit tangent vector to the track in question, considered as a point on the unit circle. The degree of a smooth map is equal modulo 2 to the number of inverse images of a regular value [4]. For a regular value we choose $(-1, 0)$, the horizontal unit vector pointing left.

First, inspection of Fig. 10 shows that each B polygon contributes exactly one to the count of inverse images of $(-1, 0)$. With notation from the definition of polygonal impossible configuration, the contribution of the B-polygons is q.

Next we will show that $(*)$ the n_i-gon A_i contributes $n_i - 2$ to this count. It will follow that the total contribution of the A-polygons is $\sum_{i=1}^{p}(n_i - 2) = N - 2p$. Since $q = N - 2p$, the grand total is the even number $2q$.

To prove $(*)$ note that an oriented n-gon in general position must fall in one of the three cases:

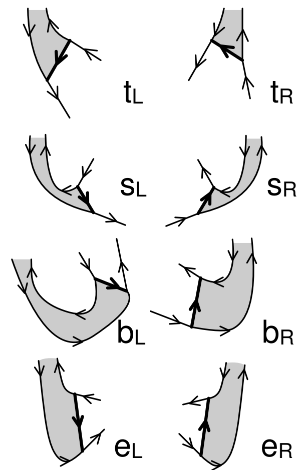

Figure 11: The eight possible relative positions of an edge in an general-position oriented polygon: $\mathbf{t}_L, \mathbf{t}_R$ top left and right, $\mathbf{s}_L, \mathbf{s}_R$ side left and right, $\mathbf{b}_L, \mathbf{b}_R$ botton left and right and $\mathbf{e}_L, \mathbf{e}_R$ top-to-bottom left and right.

   (1) $\mathbf{t}_L + \mathbf{t}_R + \mathbf{b}_L + \mathbf{b}_R + (n-4)\{\mathbf{s}_L, \mathbf{s}_R\}$
   (2) $\mathbf{e}_L + \mathbf{t}_R + \mathbf{b}_R + (n-3)\mathbf{s}_R$
   (3) $\mathbf{e}_R + \mathbf{t}_L + \mathbf{b}_L + (n-3)\mathbf{s}_L$.

   In each of these cases, the number of inverse images of $-x$ is $n-2$ (compare Figure 11.

(2) The number of self-intersection points of an immersed oriented curve in the plane, counted mod 2, is one less than its rotation number. (Because the self-intersection number mod 2 is a regular homotopy invariant [6], and because any curve with rotation number n is regularly homotopic to n turns of a spiral, with the endpoints joined [5]: a curve with $|n|-1$ intersection points). So a curve with even rotation number must have an odd number of self-intersection points.

(3) Let $\gamma_1, \ldots, \gamma_k$ be the k tracks of the path through our configuration. We know that the sum of their winding numbers is even, so an *even* number ℓ of them have odd winding number; these ℓ tracks each have even self-intersection number. The other $k-\ell$ tracks have even winding number and therefore odd self-intersection number. The sum of their self-intersection numbers is therefore congruent to $k-\ell$ and therefore to k, since ℓ is even; it follows that the sum of all the self-intersection numbers of the γ_i is congruent mod 2 to k.

(4) Finally, the self-intersection points of the path through the configuration, drawn as in Fig. 10, are of three types: those coming from the self-intersection numbers of γ_i for $i = 1, \ldots, k$, those coming from intersections between γ_i and γ_j for $i \neq j$, and those coming from the intersections of the descending arms of the B-polygons. The second and third types come in pairs. So the total number of self-intersections is congruent mod 2 to k; and this must also hold for the number of those of the first two types, which is the number of vertices of the configuration.

■

The following result is useful in constructing unicursal configurations.

Theorem 6.2. *A polygonal impossible configuration with an odd number of vertices can be adjusted, by changing the identifications in step 4 of the algorithm, to be unicursal. (This may change the genus; see Remark below.)*

Proof. uppose the configuration, displayed as in Fig. 10, is not unicursal. Then by Theorem 1 it has at least three distinct paths; label them $I, J, K, \ldots$ etc. Let us call *connectors* the parts of the B-polygons that stretch down to the A; an IJ-connector will be one with path I on the left, so leading away from the B-polygon in question, and J on the right. We will see that two of the connectors can be switched (each attaching to the A-polygon edge previously attached to the other) in such a way that the number of paths is reduced by 2. Iterating this process if necessary proves the theorem.

Assertion: There exist three paths I, J, K such that $P(IJK)$: the configuration has an IJ-connector and a KI-connector.

Proof of Assertion: Start with B_1. We can suppose without loss of generality that B_1 is incident to at least two paths, so B_1 has an IJ-connector with $I \neq J$. Now

- (*case a*) B_1 has no other paths incident, in which case it must have a JI-connector as well, or
- (*case b*) B_1 is traversed by other paths $K, L, \ldots$, so there must exist
  - (*case b1*) an IK-connector for some K
    - (*case b11*) if B_1 has a JI-connector we have $P(IKJ)$ and
    - (*case b12*) if not, B_1 must have an LI-connector for some $L \neq I$ giving $P(IJL)$.
  - or (*case b2*) a JK-connector for some K, which leads to $P(JKI)$.
- This leaves *case a*. The set of B-polygons traversed by a path or paths $K, L, \ldots$ different from I and J must be non-empty, since the configuration has at least three paths, and at least one polygon in that set must be traversed by I or J, since the configuration is connected. That polygon must contain a connector of type IK, KI, JK or KJ for some $K \neq I, K \neq J$. Since B_1 contains both an IJ and a JI, those four connectors yield, respectively, $P(IKJ)$, $P(IJK)$, $P(JKI)$, $P(JIK)$.

After the connectors have been switched, I, J and K have been welded into a single path. Compare Figure 12. Initially I goes from b to c (out of the picture) and then from d to a. Now between c and d the path also traverses the whole of J, and between a and b it traverses the whole of K. ■

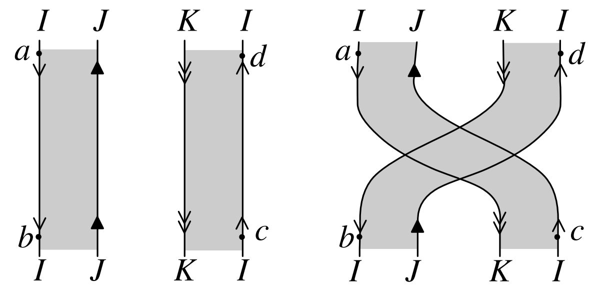

Figure 12: The relevant part of the configuration, before and after the connectors have been switched.

Remark 6.3. The (IJK) adjustment may change the genus of the resulting surface, because it also reconnects tracks of $\partial\mathcal{P}$. For example, when it is made on the 3-component configuration of Figure 2c (genus 1) it yields the uncursal configuration of Figure 2b (genus 0). ■

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