CALCULUS II
Departments of Mathematics & Applied Mathematics
The University at Stony Brook
Homework # 3

## Section 7.2

3. Compute $L_{10}$ and $R_{10}$ for $\int_1^2 x^2\,dx$, then compare these estimates with the exact value of the integral and check the bound on the magnitude of the approximation error.

*Answer.* We must subdivide the interval $[1, 2]$ into 10 subintervals of equal length. Then the step size $\triangle x$ is given by

$$\triangle x = \frac{2-1}{10} = \frac{1}{10},$$

and we insert the equally spaced points

$$1 = x_0,\ x_1 = 1 + \frac{1}{10},\ x_2 = 1 + \frac{2}{10},\ x_3 = 1 + \frac{3}{10},\ x_4 = 1 + \frac{4}{10},\ x_5 = 1 + \frac{5}{10},$$
$$x_6 = 1 + \frac{6}{10},\ x_7 = 1 + \frac{7}{10},\ x_8 = 1 + \frac{8}{10},\ x_9 = 1 + \frac{9}{10},\ x_{10} = 2$$

to partition the interval $[1, 2]$.

For the left-approximation, in each subinterval $[x_i, x_{i+1}]$ we consider the rectangle with height equal to the value of the function $f(x) = x^2$ to the left most point, fact which is illustrated in Fig. 1:

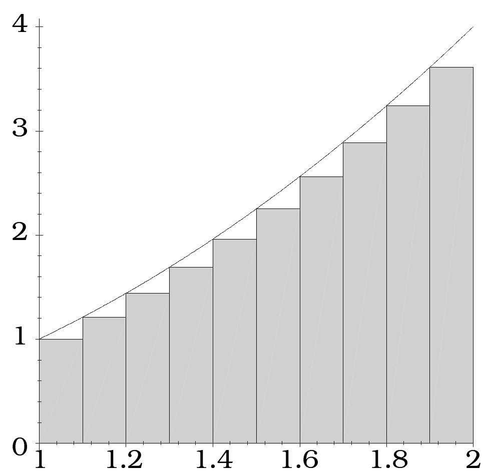


Fig. 1

We then get:

$$L_{10} = \sum_{i=0}^{9} f(x_i) \triangle x = \frac{1}{10} \sum_{i=0}^{9} \left(1 + \frac{i}{10}\right)^2 = 2.185 \, .$$

For the right approximation, in each subinterval $[x_i, x_{i+1}]$ we consider the rectangle with height equal to the value of the function $f(x) = x^2$ to the rigth most point instead:

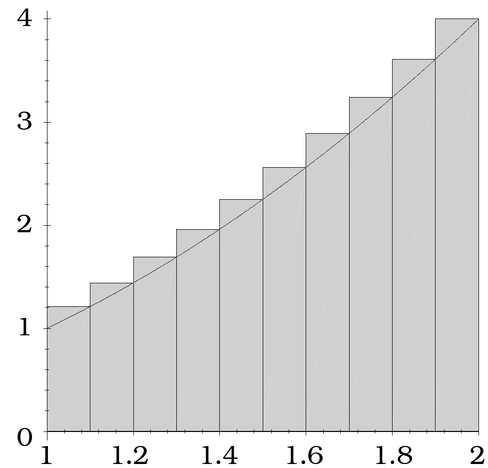

Fig. 2

We now get:

$$R_{10} = \sum_{i=1}^{10} f(x_i) \triangle x = \frac{1}{10} \sum_{i=1}^{10} \left(1 + \frac{i}{10}\right)^2 = 2.485 \, .$$

The exact value of the integral is

$$\int_1^2 x^2 \, dx = \frac{1}{3} x^3 \mid_{x=2} - \frac{1}{3} x^3 \mid_{x=1} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} = 2.333\ldots$$

We then see that $L_{10}$ is smaller than the exact value of the integral in the amount $0.148333\ldots$, while $R_{10}$ is larger than the exact value of the integral in the amount $0.151666\ldots$. That $L_{10}$ and $R_{10}$ are approximations to the integral of $x^2$ on $[1, 2]$ which are smaller and larger than the actual value of the integral can be deduced from the fact that $x^2$ is an increasing function on the interval of integration.

Theorem 2 on page 433 of the textbook says that if $K$ is such that $|f'(x)| \leq K$ for any $x$ in $[a, b]$, then if $I$ is the value of the integral, and $n$ is the number of subdivisions of $[a, b]$, we have that

$$|I - L_n| \leq \frac{K(b-a)^2}{2n} \, .$$

In our case, the derivative of $x^2$ is $2x$. On the interval $[1, 2]$ this function is bounded by 4. Hence, we get

$$|I - L_{10}| \leq \frac{4(2-1)^2}{20} = \frac{1}{5} = 0.20 \, .$$

In our particular case, the estimate given by the *general* theorem is a bit larger than the actual error.

A similar argument works for the estimation of the error $|I - R_{10}|$.

18. Give an example of a function $f$ such that $R_n$ always underestimates the value of $\int_0^5 f(x)dx$ by the maximum amount allowed by Theorem 2.

*Answer.* We know that if $f(x)$ is decreasing, then $R_n$ underestimates the value of $\int_0^5 f(x)dx$. Choose $f(x) = 5 - x$. Then

$$\int_0^5 (5-x)dx = 25 - \frac{25}{2} = \frac{25}{2},$$

while

$$R_n = \frac{5}{n}\sum_{i=1}^{n}(5 - \frac{5i}{n}) = \frac{25}{n^2}\sum_{i=1}^{n}(n-i) = \frac{25}{n^2}\sum_{j=1}^{n-1} j = \frac{25}{n^2}\frac{n(n-1)}{2} = \frac{25(n-1)}{2n}.$$

Again, Theorem 2 on page 433 of the textbook says that if $K$ is such that $|f'(x)| \leq K$ for any $x$ in $[a,b]$, then if $I$ is the value of the integral, and $n$ is the number of subdivisions of $[a,b]$, we have that

$$|I - R_n| \leq \frac{K(b-a)^2}{2n}.$$

Here we have $f'(x) = -1$ and therefore, we can choose $K = 1$. Thus,

$$|I - R_n| \leq \frac{(5-0)^2}{2n} = \frac{25}{2n}.$$

On the other hand, from the explicit calculations above, we see that

$$I - R_n = \frac{25}{2} - \frac{25(n-1)}{2n} = \frac{25}{2}\left(1 - \frac{n-1}{n}\right) = \frac{25}{2n}.$$

Thus, the error in this case is *as big* as the estimate given by Theorem 2.

20 c) For which of the first 6 integrals does $T_{10}$ make no approximation error?

*Answer.* The trapezoidal approximation makes a larger error the more concave the graph of the function is, making no error at all if the graph is *flat*. This happens for linear functions. Thus, $T_{10}$ makes no error for $\int_2^3 1\,dx$ and for $\int_1^3 x\,dx$.

20 d) Which integrals does $T_{10}$ underestimate?

*Answer.* If the graph of $f$ is concave-down in the domain of integration, $T_n$ underestimates the integral of $f$ on that interval. Since the concavity is determined by the sign of the second derivative, calculating the second derivatives of the integrands we see that $T_{10}$ underestimates $\int_1^4 \sqrt{x}\,dx$ and $\int_2^3 \sin x\,dx$.

**Section 7.3**

1. Let $I = \int_0^1 e^{x^2}\,dx$.

a) Compute $M_2$ and $T_2$ by hand.

*Answer.* The function $f(x)$ is in this case $f(x) = e^{x^2}$. We subdivide the interval $[0,1]$ into $n = 2$ subintervals of equal length, $[x_0, x_1]$ and $[x_1, x_2]$ where $x_0 = 0$, $x_1 = 1/2$ and $x_2 = 1$, respectively. Then $\triangle x = (1-0)/2 = 1/2$ and

$$\begin{aligned} M_2 &= \textstyle\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\triangle x \\ &= \tfrac{1}{2}\textstyle\sum_{i=1}^{2} f\left(\frac{x_{i-1}+x_i}{2}\right) \\ &= \tfrac{1}{2}\left[f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_1+x_2}{2}\right)\right] \\ &= \tfrac{1}{2}\left[f(\tfrac{1}{4})+f(\tfrac{3}{4})\right] \\ &= \tfrac{1}{2}\left[e^{\frac{1}{16}}+e^{\frac{9}{16}}\right] \\ &= 0.5\,[1.064494459+1.755054657] \\ &= 1.409774558\,. \end{aligned}$$

Similarly,

$$\begin{aligned} T_2 &= \textstyle\sum_{i=1}^{n} \frac{f(x_{i-1})+f(x_i)}{2}\triangle x \\ &= \tfrac{1}{2}\textstyle\sum_{i=1}^{2} \frac{f(x_{i-1})+f(x_i)}{2} \\ &= \tfrac{1}{4}\,[f(x_0)+2f(x_1)+f(x_2)] \\ &= \tfrac{1}{4}\,[1.000000+2\cdot 1.284025417+2.718281828] \\ &= 0.25\cdot 6.286332662 \\ &= 1.571583166\,. \end{aligned}$$

b) Compute $L_{10}$, $R_{10}$, $M_{10}$, and $T_{10}$. Which of these underestimates the exact value of $I$.

*Answer.* Since the function $f(x) = e^{x^2}$ increases and it is concave-up on $[0,1]$, both $L_{10}$ and $M_{10}$ underestimate the exact value of the integral. Similarly, $R_{10}$ and $T_{10}$ overestimate the value. We have:

$$\begin{aligned} L_{10} &= \tfrac{1}{10}\textstyle\sum_{i=0}^{9} e^{(\frac{i}{10})^2} = \tfrac{1}{10}\sum_{i=0}^{9} e^{\frac{i^2}{100}} = 1.381260601\,, \\ R_{10} &= \tfrac{1}{10}\textstyle\sum_{i=1}^{10} e^{(\frac{i}{10})^2} = \tfrac{1}{10}\sum_{i=1}^{10} e^{\frac{i^2}{100}} = 1.553088784\,, \\ T_{10} &= \tfrac{1}{20}\textstyle\sum_{i=1}^{n}\left(e^{\frac{(i-1)^2}{100}}+e^{\frac{i^2}{100}}\right) = 1.467174693\,, \\ M_{10} &= \tfrac{1}{10}\textstyle\sum_{i=1}^{n} e^{\frac{i}{10}-\frac{1}{20}} = 1.460393091 \end{aligned}$$

c) Use Theorem 3 to find $n$ such that the error $|I - M_n| \le 0.0005$.

*Answer.* Since $f''(x) = (4x^2+2)e^{x^2}$, on $[0,1]$ this is bounded by its value at $x = 1$, that is to say, by $6e$. Then,

$$|I - M_n| \le \frac{6e(1-0)^3}{24n^2} = \frac{e}{4n^2}\,.$$

Since we want the error to be less than 0.0005, we must choose $n$ such that

$$\frac{e}{4n^2} \leq 0.0005\,.$$

Solving for $n$, we obtain

$$n \geq \sqrt{e/(4 \cdot 0.0005)} > 36.866\,.$$

Hence, we just need to choose $n = 37$ or greater to ensure the desired accuracy.

3. Suppose that $f''(x) = \frac{e^x \cos x}{1+x^2}$. Find an integer $n$ such that $T_n$ approximates $\int_0^5 f(x)dx$ within 0.001.

*Answer.* According to Theorem 3 on page 440 of the textbook, if $K$ is an upper bound for $|f''(x)|$ on the interval $[a, b]$, then if $I$ is the value of the integral and $T_n$ is the trapezoidal approximation with $n$ subdivisions of $[a, b]$, we have that

$$|I - T_n| \leq \frac{K(b-a)^3}{12n^2}\,.$$

In this problem, we know that $f''(x) = \frac{e^x \cos x}{1+x^2}$. The absolute value of this function has a maximum in between $x = 3$ and $x = 4$, which is not greater than 2.5 (in fact, it is not greater than 2.4). Hence,

$$|f''(x)| \leq 2.5\,,$$

and we have that

$$|I - T_n| \leq \frac{2.5(5-0)^3}{12n^2}\,.$$

We want the error to be no larger than 0.001. So we choose $n$ such that

$$\frac{2.5(5-0)^3}{12n^2} = \frac{26.41666}{n^2} \leq 0.001\,.$$

Solving for $n$, we see that

$$n \geq 161.37\,.$$

Thus, we may choose $n = 162$ or greater to ensure an accuracy better than 0.001.

14. Using left, right, midpoint and trapezoidal rules, we obtained approximations of $\int_a^b f(x)dx$ using the same number of subdivisions of the interval $[a, b]$. These approximations were 8.52974, 9.71090, 9.74890 and 11.04407. Which rule produce which estimate?

*Answer.* The function depicted in the graph is increasing and concave-up. Hence, both left and midpoint approximation underestimates the interval, and right and trapezoidal overestimate it. But we know that the midpoint approximation is a better approximation to the integral than the left approximation, while the trapezoidal is better than the right approximation. Hence, we have

$$L_n < M_n < I < T_n < R_n\,.$$

Thus, $L_n$ is the smallest of the given approximations, while $R_n$ is the largest. $M_n$ is the second smallest. So we get:

$$L_n = 8.52974\,, \quad M_n = 9.71090\,, \quad T_n = 9.74890\,, \quad R_n = 11.04407\,.$$