Trigonometry for Adults

Tony Phillips, November 2014

Students need help in making the transition from high-school trigonometry (based on right-triangle trig, an awkward conversion to the circle – “quadrants” – and measurement mostly in degrees) to the trigonometry used in college and beyond (emphasis on sin, cos, etc. as functions of a variable measured in radians). Here’s what we can do to make it easier for them.

I. Whenever a trigonometry problem arises, put this diagram on the board. Encourage your students to copy this diagram and to learn to reproduce it themselves.

![Trigonometric Circle Diagram]

Figure 1: The “trigonometric circle.” When $t$ is measured counterclockwise along the unit circle starting from 1 on the horizontal axis, $\cos t$ and $\sin t$ are the $x$ and $y$ coordinates of $t$.

If you and your students work from the trigonometric circle, you can’t go wrong. Many important identities, e.g. $\sin^2 t + \cos^2 t = 1$, $\sin(t + \pi) = -\sin t$, $\cos(t + \pi) = -\cos t$ can be deduced or reliably checked using this diagram.
II. Radian measure and degrees.

The trigonometric circle can also help clarify the relation between degree-measure and radian-measure of an angle.

![Diagram of trigonometric circle with radian measure](image)

Figure 2: The radian measure of an angle is determined by drawing it at the center of the trigonometric circle, with the leading edge along the positive x-axis, and reading off the length of the intercepted arc.

You can explain to your students that degree-measurement of angles is a cultural artifact dating back to the Old Babylonians (c. 2000 BC), who used a base-60 representation of numbers. Radian measure, geometrically more intrinsic, is fundamental to the use of trigonometric functions in calculus. For example, the approximation \( \sin t \approx t \), which works very well when \(|t|\) is small, only works when \( t \) is measured in radians.

The trigonometric circle gives a reliable method for converting degrees to radians, and for estimating how big an angle is from its radian measure. For example, since 60° is one-sixth of the “total angle” 360°, the corresponding radian measure will be one-sixth of the complete circumference of the unit circle, i.e. \( 2\pi/6 = \pi/3 \). On the other hand, since \( \pi/2 \) is 1.570..., radian measure 1, which is about 2/3 of that, corresponds to about \((2/3) \times 90° = 60°\).
III. Extend the use of the trigonometric circle to tan.

Figure 3: The trigonometric circle gives a reliable definition of the tangent function. Extend the radius line from the center to the end of the arc $t$ (in either direction) until it intersects the “tangents” axis: the oriented line tangent to the circle at $x = 1$. The height of that intersection point is $\tan t$. [There are two congruent right triangles in the figure, one with legs $\cos t$ and $\sin t$, the other with corresponding legs 1 and $\sin t/\cos t = \tan t$].

This representation allows reliable retrieval of facts about the function $\tan t$. For example, $\lim_{t \to \pi/2} \tan t = +\infty$, $\lim_{t \to -\pi/2} \tan t = -\infty$, $\tan(t + \pi) = \tan t$. 
IV. Connecting with right-triangle trigonometry.

It is essential to maintain the link to the right-triangle-trigonometry definitions of sin \( t \), cos \( t \), tan \( t \) when \( t \) is one of the acute angles in a right triangle. Often in calculus right-angle-trigonometry is used to set up the problem, and then knowledge of the behavior of sin, cos, tan as functions is used to solve it.

![Diagram of right triangle showing sin, cos, and tan](image)

Figure 4: When \( t \) is the radian measure of one of the acute angles of a right triangle, then sin \( t \) is the ratio of the lengths: (opposite side)/(hypothenuse), cos \( t \) is the ratio of the lengths: (adjacent side)/(hypothenuse), and tan \( t \) is the ratio of the lengths: (opposite side)/(adjacent side).

In particular the values of the trigonometric functions at the canonical angles \( \frac{\pi}{3} \), \( \frac{\pi}{4} \), \( \frac{\pi}{6} \) can be rapidly and reliably derived from elementary geometry and Pythagoras' Theorem. You and all your students should know how to recalculate these values when necessary.

![Diagram of right-angled triangles](image)

Figure 5: Right isosceles triangle: \( \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \), \( \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \) also, giving \( \tan \frac{\pi}{4} = 1 \). Equilateral triangle: \( h = \sqrt{a^2 - (a/2)^2} = \sqrt{3}a^2/4 = \frac{a\sqrt{3}}{2} \). So \( \sin \frac{\pi}{3} = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \), and \( \sin \frac{\pi}{6} = \cos \frac{\pi}{3} = \frac{1}{2} \). This gives \( \tan \frac{\pi}{3} = \sqrt{3} \) and \( \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \).
V. Graphing $\sin t$.

When sketching the graph of $\sin t$ (or $\cos t$) try to remember that $\pi$ is a little more than 3, whereas $|\sin t| \leq 1$, and to scale your graph realistically. This will help your students grasp and remember the geometric implications of $\sin'(0) = \cos(0) = 1$ and $\sin'(\pi) = \cos(\pi) = -1$, as well as $\int_0^\pi \sin t \, dt = 2$, etc.

Figure 6: A sketch of the graph of $\sin t$ trying to respect the scale.