MAT511 homework, due Oct 21, 2009

(1) Prove carefully by induction that the binomial coefficients
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
satisfy
$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k+1}.$$
(Remember the convention $0! = 1! = 1$.

(2) Current (non-vanity) NYS license plates have the format "ABC 1234" with letters and numbers. How many possible different plates of this format can there be? Suppose each of the 7 positions could hold either a letter or a number. Then how many could there be?

(3) Let A and B be nonempty sets. Prove that $A \times B = B \times A$ if and only if $A = B$. What if one of A or B is empty?

(4) For each of the relations below, indicate whether it is reflexive, symmetric, or transitive. Justify your answer.
  - (a) $\leq$ on the set $\mathbf{N}$.
  - (b) $\perp = \{(l, m)$ such that l and m are lines, with l perpendicular to $m\}$.
  - (c) $\sim$ on $\mathbf{R} \times \mathbf{R}$, where $(x, y) \sim (z, w)$ if $x + z \leq y + w$.
  - (d) $\smile$ on $\mathbf{R} \times \mathbf{R}$, where $(x, y) \smile (z, w)$ if $x + y \leq z + w$.
  - (e) $\square$ on $\mathbf{R} \times \mathbf{R}$, where $(x, y) \square (z, w)$ if $x + z = y + w$.

(5) Prove that if R is a symmetric, transitive relation on a set A, and the domain of R is A, then R is reflexive on A.

(6) Consider the relations $\sim$ and $\square$ on $\mathbf{N}$ defined by $x \sim y$ iff $x + y$ is even, and $x \square y$ iff $x + y$ is a multiple of 3. Prove that $\sim$ is an equivalence relation, and that $\square$ is not.

(7) For each $a \in \mathbf{R}$, let $P_a = \{(x, y) \in \mathbf{R} \times \mathbf{R}$ such that $y = a - x^2\}$.
  - (a) Sketch the graph of P_{-2}, P_0, and P_1.
  - (b) Prove that $\{P_a$ such that $a \in \mathbf{R}\}$ forms a partition of $\mathbf{R} \times \mathbf{R}$.
  - (c) Describe the equivalence relation associated with this partition.