MAT 342 Fall 2010
Review for Final Examination

Material on first two review sheets plus the following.

Section 42. Review the calculation of $\int_C z^n$ (n an integer, positive or negative; C the unit circle about 0 (pp. 134-135): $= 2\pi i$ when $n = -1$, 0 otherwise.

Section 43. Understand what it says (it stands to reason) and be able to use the inequality

$$|\int_C f(z)\ dz| \leq ML$$

where M is the maximum value of $|f(z)|$ on the set C, and L is the length of C. Exercises 1, 2, 3.

Section 44. Understand and be able to use the main Theorem (f has antiderivative $\Leftrightarrow$ contour integrals of f only depend on endpoints $\Leftrightarrow$ integrals of f along a closed contour are 0). Exercises on p. 149: 1, 2, 3.

Section 46. Understand the statement of the Cauchy-Goursat theorem: *If f is analytic on and inside a simple closed contour C, then $\int_C f(z)dz = 0$.* Be able to apply this theorem!

Section 49. Understand why the Corollary (p.159) works and how to use it. Important Example p. 1160:

$$\int_C \frac{dz}{z} = 2\pi i$$

for any simple closed curve going *counterclockwise* around the origin. Exercises 1, 2, 3.

Section 50. Understand how to use the Cauchy Integral Formula:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s - z}\ ds.$$

Know all the hypotheses: f analytic on and inside C, integration is counterclockwise, z is in the interior of C. Understand the Example p. 164; it is simple but typical.

Section 51. Understand and be able to use the Cauchy-type formulas for $f'(z)$ and $f''(z)$. Do as many of the exercises on pp. 170-171-172 as you can.

Sections 53, 54. These are of great theoretical importance but not directly relevant to our applications. (Liouville's Theorem, Fundamental Theorem of Algebra, Maximum Modulus Principle).

Sections 55, 56. Understand that convergence of a complex series is equivalent to separate convergence of its real and imaginary parts. Understand that a series converges to a sum if and only if the sequence of remainders tends to zero, and use this as in the Example p.187 to show

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{for} \quad |z| < 1.$$

Sections 57, 58. Understand Taylor's Theorem and why it is different from usual convergence theorems (which depend, like the ratio test, on analysis of the coefficients in the series): if f is analytic in a disc $|z - z_0| < r$ then the Taylor expansion

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

about z_0 converges for every z in the disc. Consequence: the Taylor series for an entire function converges everywhere! All the examples in Section 59 are worth reviewing.

Section 60. Laurent Series. Compare with Taylor's theorem: If f is analytic in an annulus $r < |z - z_0| < R$ then the Laurent series for f

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z - z_0)^n}$$

converges at every point of the annulus. Here the coefficients a_n, b_n can't be defined at z_0 as in Taylor's formula, but are calculated from values along a contour C going once counterclockwise around the annulus, by

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}, \qquad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}}.$$

(Notice that if f is actually analytic in the *disc* $|z - z_0| < R$ then the a_n integral is equal to $\frac{f^{(n)}(z_0)}{n!}$, the coefficient from the Taylor series, and the b_n integrals have an analytic integrand and are therefore 0.)

Section 62. Examples of Laurent series. Notice that the coefficients are hardly ever calculated using the integral formulas. Examples 1 and 2 should both be reviewed.

Section 68-69. Understand what it means for z_0 to be an *isolated singular point* of f, and that then f has a Laurent series valid in an annular neighborhood (punctured disc) $0 < |z - z_0| < R$. The residue of f at z_0 is then the coefficient b_1 in this Laurent series. Note that then

$$\int_C f(z)\,dz = 2\pi i b_1$$

for any counterclockwise contour in the punctured disc. This can be used to calculate integrals, as in Examples 1, 2, 3 on pp.232-233.

Sections 70,71. "Cauchy's Residue Theorem" follows directly from the last equation and things we know about contour integrals. Understand that if the contour C encloses *all* the singularities of f, then the problem can be turned inside-out, and

$$\int_C f(z)\ dz = 2\pi i \text{Res}_{z=0}[\frac{1}{z^2} f(\frac{1}{z})].$$

Exercise 3a is typical.

Section 72. This section is mainly of theoretical interest, but it shows clearly how the b-section of the Laurent series about z_0 characterizes the nature of the singularity of f at z_0.

Sections 73-74. Continuing, we focus on poles of order m ($b_m \neq 0$, but all higher-indexed bs are 0). Then $f(z) = \phi(z)/(z - z_0)^m$, ϕ analytic at z_0, and there is a convenient expression for the residue:

$$\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

The 5 Examples should be reviewed.

Section 76. Has a handy formula for the residue of a quotient of analytic functions, when the quotient has a simple pole at z_0: If $f(z) = p(z)/q(z)$, $p(z_0) \neq 0$, $q(z_0) = 0$, $q'(z_0) \neq 0$ then its residue at z_0 is $p(z_0)/q'(z_0)$. Review Examples 2, 3, 4.

Sections 78-79. Go over the examples carefully. The example starting on page 254 is basic. We have done several exampls of this type, and you should be comfortable with it and exercises 1-5 on p. 267.