Midterm 1 Solutions

Note that there are different forms of this test; yours may be slightly different from this one.

1. (a) (15 points) What are the 4 fourth roots of $-9$?

Write $-9$ as $9e^{i\pi}$. Then the rule for $n$-th roots gives the four roots as

$$\sqrt[4]{9e^{i\pi}} = \sqrt[4]{3e^{i\frac{\pi + 2k\pi}{4}}}$$

$k = 0, 1, 2, 3$.

(b) (15 points) Write $z^4 + 9$ as $(z - r_1)(z - r_2)(z - r_3)(z - r_4)$.

The four roots calculated above are:

$$r_1 = \sqrt{3}e^{i\frac{\pi}{4}} = \sqrt{3}\frac{1+i\sqrt{2}}{2}$$
$$r_2 = \sqrt{3}e^{i\frac{\pi + 2\pi}{4}} = \sqrt{3}\frac{1+i}{\sqrt{2}}$$
$$r_3 = \sqrt{3}e^{i\frac{\pi + 4\pi}{4}} = \sqrt{3}\frac{-1+i\sqrt{2}}{2}$$
$$r_4 = \sqrt{3}e^{i\frac{\pi + 6\pi}{4}} = \sqrt{3}\frac{-1-i}{\sqrt{2}}$$

This gives

$$z^4 + 9 = (z - \sqrt{3}\frac{1+i}{\sqrt{2}})(z - \sqrt{3}\frac{-1+i}{\sqrt{2}})(z - \sqrt{3}\frac{-1-i}{\sqrt{2}})(z - \sqrt{3}\frac{1-i}{\sqrt{2}}).$$

(c) (15 points) Use the fact that the complex roots of a polynomial with real coefficients come in complex conjugate pairs to write $z^4 + 9$ as a product of two quadratic polynomials with real coefficients.

In this case $r_1$ and $r_4$ are complex conjugates, as are $r_2$ and $r_3$. To shorten notation, notice that $(z - a)(z - \overline{a}) = z^2 - 2\Re(a) + |a|^2$, where $\Re(a)$ is the real part of $a$. So:

$$(z - r_1)(z - r_4) = z^2 - 2\sqrt{3}\frac{1}{\sqrt{2}}z + 3 = z^2 - \sqrt{6}z + 3$$

and

$$(z - r_2)(z - r_3) = z^2 - 2\sqrt{3}\frac{-1}{\sqrt{2}}z + 3 = z^2 + \sqrt{6}z + 3$$
so finally
\[ z^4 + 9 = (z^2 - \sqrt{6}z + 3)(z^2 + \sqrt{6}z + 3). \]

An alternative method was to write
\[ z^4 + 9 = (z^2 + 3i)(z^2 - 3i) \]
and
\[ z^2 + 3i = (z - i\sqrt{3}i)(z + i\sqrt{3}i); \quad z^2 - 3i = (z + \sqrt{3}i)(z - \sqrt{3}i) \]
to get
\[ z^4 + 9 = (z - i\sqrt{3}i)(z + i\sqrt{3}i)(z + \sqrt{3}i)(z - \sqrt{3}i) \]
Here it’s not obvious which roots are complex conjugates: best to work it out with \( \sqrt{3i} = \sqrt{3}^{1+i} \).

2. (a) (15 points) What is the image of the line \( \Im(z) = 1 \) [i.e. \( \{x+iy|y = 1\} \)] under the mapping \( w = z^2 \)?

The mapping \( w = z^2 \) takes \((x, y)\) to \((u = x^2 - y^2, v = 2xy)\). So the line \( \Im(z) = 1 \) goes to \((u = x^2 - 1, v = 2x)\). [This is also true of the line \( \Im(z) = -1 \) given on some of the forms of the test]. The image of the line is therefore the parabola \((u = x^2 - 1, v = 2x)\), or \( u = (\frac{v}{2})^2 - 1 \).

(b) (15 points) Sketch the image of the half-plane \( \Im(z) \geq 1 \) under the mapping \( w = z^2 \).

The mapping \( w = z^2 \) takes each line \( \Im(z) = c \) into a parabola; when \( c = 0 \) this is the degenerate parabola represented by the positive \( u \)-axis covered twice. As \( 0 \leq c \leq 1 \) these parabolas fill the shaded region in the picture here, the “inside” of the parabola \( u = (\frac{v}{2})^2 - 1 \). As \( 1 \leq c \leq \infty \) the parabolas fill in the outside of the shaded area. So the closed upper half-plane \( \Im(z) \geq 0 \) maps onto the entire \((u, v)\)-plane, with the positive \( u \)-axis covered twice.

Since \((-z)^2 = z^2\) the same thing happens for negative imaginary values: The region \(-1 \leq y \leq 0\) maps to the shaded area, and the region \(-\infty \leq y \leq -1\) fills in the outside of the shaded area.

Depending on the form of the test you had, the answers were:
• \( \Im(y) \geq 1 \) maps to the outside of the shaded region.
• \( \Im(y) \geq -1 \) covers the whole plane. (Shaded region gets covered twice).
• \( \Im(y) \leq 1 \) covers the whole plane. (Shaded region gets covered twice).
• \( \Im(y) \leq -1 \) maps to the outside of the shaded region.

Figure 1: The parabola \( u = (v/2)^2 - 1 \) is the image of the line \( \Im(z) = 1 \) [and also of the line \( \Im(z) = -1 \)].

3. (a) (15 points) Show carefully by an \( \epsilon, \delta \) argument that

\[
\lim_{z \to a} \frac{f(z)}{g(z)} = 0
\]

if \( \lim_{z \to a} f(z) = 0 \) and if there exists a pair of positive numbers \( \delta_0, M \) such that \( |z - a| < \delta_0 \) implies \( |g(z)| \geq M \).
Part (a) was almost identical to Problem 9 page 54, which we went over twice in class:

Given \( \epsilon \):

Since \( \lim_{z \to a} f(z) = 0 \) there is a \( \delta_1 \) such that \( |z-a| < \delta_1 \) guarantees \( |f(z)| < \epsilon M \).

Now take \( \delta = \min(\delta_0, \delta_1) \). If \( |z-a| < \delta \), then \( |f(z)| < \epsilon M \) and \( |g(z)| \geq M \); hence

\[
\left| \frac{f(z)}{g(z)} \right| = \frac{|f(z)|}{|g(z)|} < \frac{\epsilon M}{M} = \epsilon.
\]

(b) (10 points) Apply this to prove that

\[
\lim_{z \to 0} \frac{z}{2 + \frac{\bar{z}}{z}} = 0.
\]

Since \( \lim z \to 0 z = 0 \), it is enough by part (a) to find a \( \delta_0 \) and an \( M \) that work for the denominator \( 2 + \frac{\bar{z}}{z} \). (You need something like this, because \( \lim_{z \to 0} \frac{z}{\bar{z}} \) does not exist). The triangle inequality (“backwards”) gives

\[
|2 + \frac{\bar{z}}{z}| \geq 2 - \left| \frac{\bar{z}}{z} \right| = 2 - \frac{\left| \bar{z} \right|}{\left| z \right|} = 2 - 1 = 1.
\]

So any positive number works for \( \delta_0 \) with \( M = 1 \).