Some limit problems: solutions.

The method is always the same. The problem is of the form

$$\lim_{z \to a} f(z) = L.$$

Unraveling the definition of limit, this means that given any $\epsilon > 0$ we can produce a $\delta > 0$ so that $|f(z) - L| < \epsilon$ whenever $|z - a| < \delta$.

We proceed as follows: we work backwards, by setting $z = a + d$ and looking at how far $f(a + d)$ is from L. We calculate that distance in terms of $|d| = \delta$ and then find a way to bound δ to make that distance less than ϵ.

1.

$$\lim_{z \to i} \frac{1}{(z+i)^2} = \frac{-1}{4}$$

Solution: Here the function whose limit we are studying is $f(z) = \frac{1}{(z+i)^2}$. Given $\epsilon > 0$ we need to produce a δ so that $|f(z) - \frac{1}{4}| < \epsilon$ whenever $|z - i| < \delta$. We work backwards, by setting $z = i + d$ and looking at how far $f(z)$ is from $\frac{1}{4}$. Then we will calculate how to bound $|d| = \delta$ to make that distance less than ϵ.

$$f(i + d) = \frac{1}{(2i + d)^2}.$$

$$f(i + d) - \frac{-1}{4} = \frac{4 + (2i + d)^2}{4(2i + d)^2} = \frac{4id + d^2}{4(2i + d)^2}.$$

RULE: to control a fraction, you need an upper bound for the numerator and a *lower* bound for the denominator.

Control of denominator: We work on the $(2i + d)^2$ in the denominator by running the triangle inequality backwards: $2i = (2i + d) - d$ so $|2i| \le |2i + d| + |d|$ and $|2i + d| \ge |2i| - |d| = 2 - \delta$.

If $\delta < 1$ then $2 - \delta > 1$ and the denominator $4(2i + d)^2$, in absolute value, will satisfy

$$|4(2i + d)^2| = 4|2i + d|^2 \ge 4(2 - \delta)^2 > 4.$$

Control of numerator: By the triangle inequality,

$$|4id + d^2| \le 4\delta + \delta^2.$$

Since when $\delta < 1$ the denominator is > 4, we can make the whole fraction less than ϵ by further shrinking δ to make the numerator $< 4\epsilon$. We can do that by choosing $\delta < \epsilon/2$, because then $4\delta < 2\epsilon$ and $\delta^2 < \epsilon/2$ (since we have already required $\delta < 1$) so their sum is $< 2\epsilon + \epsilon/2 < 4\epsilon$. Putting it all together:

If $|d| = \delta < \min(\epsilon/2, 1)$ then $|f(i+d) - (-1/4)| < \epsilon$.

2.

$$\lim_{z \to -i} (\overline{z}^2 - z) = i - 1$$

Solution: Set $z = -i + d$; then $\overline{z} = i + \overline{d}$ and

$$\overline{z}^2 - z - (i-1) = i^2 + 2i\overline{d} + \overline{d}^2 + i - d - i + 1 = \overline{d}^2 + 2i\overline{d} - d.$$

In absolute value,

$$|\overline{z}^2 - z - (i-1)| = |\overline{d}^2 + 2i\overline{d} - d)| \leq \delta^2 + 2\delta + \delta = \delta^2 + 3\delta$$

since d and $\overline{d}$ have the same absolute value.

So choosing $\delta < \min(\epsilon/6, 3)$ makes

$$\delta^2 + 3\delta < \epsilon/2 + \epsilon/2 = \epsilon.$$

3.

$$\lim_{z \to i} \frac{z^3 + i}{z - i} = -3$$

Solution: Set $z = i + d$; then since $(i+d)^3 = -i - 3d + 3id^2 + d^3$, we have

$$\frac{z^3 + i}{z - i} = \frac{-i - 3d + 3id^2 + d^3 + i}{i + d - i} = \frac{-3d + 3id^2 + d^3}{d} = -3 + 3id + d^2.$$

The distance from this point to -3 is

$$|-3 + 3id + d^2 + 3| = |3id + d^2| \leq 3\delta + \delta^2.$$

We can make this distance smaller than ϵ by choosing $\delta < \min(\epsilon/6, 3)$. Then

$$3\delta + \delta^2 < 3(\epsilon/6) + 3 \cdot \epsilon/6 = \epsilon.$$