MAT 342
Applied Complex Analysis
Midterm 2

April 12, 2007

SOLUTIONS

1. (a) (12 points) Using the definition $e^z = e^x \cos y + ie^x \sin y$, where $z = x + iy$, show that the function $f(z) = e^z$ is analytic.

   SOLUTION: It is enough, writing e^z as $u(x,y) + iv(x,y)$, to check that u and v are differentiable and satisfy the Cauchy-Riemann equations. Here $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$. $u_x = e^x \cos y = v_y$ and $u_y = -e^x \sin y = -v_x$.

   (b) (12 points) Taking $e^z = e^x \cos y + ie^x \sin y$ as your definition, show that

$$\frac{d}{dz} e^z = e^z.$$

   SOLUTION: Using $f'(z) = u_x + iv_x$, with $u_x = e^x \cos y$ and $v_x = e^x \sin y$ gives $(d/dz)e^z = e^x \cos y + ie^x \sin y = e^z$.

2. (a) (14 points) Evaluate

$$\int_C \frac{dz}{z^2 + 2z + 4}$$

   where C is the circle of radius 2 about $2i$, traversed counterclockwise.

   SOLUTION: By the quadratic formula, the roots of $z^2 + 2z + 4$ are $z = -1 \pm i\sqrt{3}$. The root $-1 + i\sqrt{3}$ is inside the contour. Write $z^2 + 2z + 4 = (z + 1 + i\sqrt{3})(z + 1 - i\sqrt{3})$, and use $\dfrac{1}{(z + 1 + i\sqrt{3})}$ as your $f(z)$ and $(z + 1 - i\sqrt{3})$ as your $z - z_0$ in Cauchy's Integral Formula $\displaystyle\int_C \frac{f(z)\ dz}{z - z_0} = 2\pi i f(z_0)$. In this case $f(z_0) = \dfrac{1}{-1 + i\sqrt{3} + 1 + i\sqrt{3}} = \dfrac{1}{2i\sqrt{3}}$. The integral is then $2\pi i f(z_0) = \frac{\pi}{\sqrt{3}}$.

   (b) (12 points) Show that if C_R is the semicircle $|z| = R$, $\Im(z) \geq 0$ ($\Im(z)$ is the imaginary part of z), then

$$\lim_{R\to\infty} \int_{C_R} \frac{dz}{z^2 + 2z + 4} = 0.$$

Hint: you may want to use the triangle inequality in the form $|a+b+c| \geq |a| - |b| - |c|$.

SOLUTION: We use the inequality $|\int_C f(z)dz| \leq ML$, where L is the length of C and M is $\max_{z\in C}|f(z)|$. Using the triangle inequality as suggessted, we have

$$|\frac{1}{z^2+2z+4}| \leq \frac{1}{|z^2|-|2z|-4} = \frac{1}{R^2-2R-4}$$

on the semicircle of radius R, so we can take this number as M. $L = \pi R$, so $ML = \frac{\pi R}{R^2-2R-4}$, and $\lim_{R\to\infty} ML = 0$

(c) (14 points) Calculate $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+4}$. (If you can do this without complex analysis, that's fine too).

The easiest way to do this is to write

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+4} = \int_{-\infty}^{\infty} \frac{dz}{z^2+2z+4} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dz}{z^2+2z+4}.$$

Now $\int_{-R}^{R} \frac{dz}{z^2+2z+4} + \int_{C_R} \frac{dz}{z^2+2z+4}$ is the integral around a contour containing the root $-1+i\sqrt{3}$ of the denominator, so as in part (a), the sum of those two integrals is $\frac{\pi}{\sqrt{3}}$, no matter what R is. In the limit as $R \to \infty$ the first integral is $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+4}$ and the second integral is 0.

This part could also be solved without complex analysis by completing the square and writing the denominator as $(z+1)^2+3$ Then the substitution $u = z+1$ leads to $\int_{-\infty}^{\infty} \frac{du}{u^2+3}$. The substitution $u = v\sqrt{3}, du = dv\sqrt{3}$ leads to

$$\frac{\sqrt{3}}{3}\int_{-\infty}^{\infty} \frac{dv}{v^2+1} = \frac{\sqrt{3}}{3} \arctan v|_{-\infty}^{\infty} = \frac{\pi}{\sqrt{3}}.$$

3. (12 points) Evaluate $\int_C \frac{e^{3z}}{z^2}\, dz$, where C is the circle $|z| = 1$, traversed counterclockwise.

SOLUTION: Here use Cauchy's Formula for $2\pi i f'(z_0)$, with $z_0 = 0$ and $f(z) = e^{3z}$. In this case $f'(z_0) = 3e^0 = 3$, so the integral is $6\pi i$.

4. (a) (12 points) Show that

$$\int_C \frac{dz}{z^4} = 0,$$

where C is the circle $|z| = 1$, traversed counterclockwise, by direct calculation or by quoting an *appropriate* theorem.

SOLUTION: There were various ways of doing this, but you could NOT apply Cauchy's Theorem directly, because $\frac{1}{z^4}$ is not analytic at 0. On the other hand $\frac{1}{z^4}$ is the derivative of $-1/3z^3$, so its integral around any closed path is zero.
Alternately, you could do a direct calculation: parametrize the circle by $e^{i\theta}, 0 \leq \theta \leq 2\pi$. Then $\frac{1}{z^4} = e^{-4i\theta}$ and $dz = ie^{i\theta}\ d\theta$, so

$$\int_C \frac{dz}{z^4} = \int_0^{2\pi} ie^{-3i\theta}\ d\theta = (-1/3i)ie^{-3i\theta}|_0^{2\pi} = (-1/3)(1-1) = 0.$$

And there were other ways.

(b) (12 points) Show that

$$\int_S \frac{dz}{z^4} = 0,$$

where S is the boundary of a pentagon with vertices at $3i, \pm 3, \pm 2 - 2i$, by a method of your choice.

SOLUTION: If you used the "anti-derivative" argument for (a) you can use it again here. If you used the direct calculation, you need to remark that $\frac{1}{z^4}$ is analytic in the space between the pentagon and the circle, so the integrals are the same, namely 0.