My NAME is:

MAT 342
Applied Complex Analysis
Final Examination

May 15, 2007

Hand back this text with your bluebook, your name on each.
Show all your work in your bluebook! Total score = 200

You may quote theorems we have learned this semester, but for full credit you must state them carefully, check that the hypotheses are satisfied, and (of course) use them appropriately.

Some theorems with complicated formulas are reproduced on a sheet attached to this examination.

1. (20 points) Show directly from the definition of complex derivative (i.e. do *not* invoke the Cauchy-Riemann equations) that the function $f(z) = \overline{z}^2$ is not differentiable anywhere except at $z = 0$.

2. (a) (10 points) Draw the image of the square $0 \leq x \leq 1, \quad 0 \leq y \leq 1$ under the mapping $A(z) = iz - 1$.

   (b) (10 points) Show that the mapping A satisfies $A^4 = I$, i.e. $A(A(A(A(z)))) = z$ for any z.

3. (10 points) Let $\mathcal{L}\text{og}$ represent a branch of the complex logarithm function; so

$$e^{\mathcal{L}\text{og}(z)} = z.$$

Show that at a point z where $\mathcal{L}\text{og}$ is analytic, its derivative satisfies

$$\mathcal{L}\text{og}'(z) = \frac{1}{z}.$$

4. (10 points) Use the parametrization $z = e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$ to directly evaluate the contour integral

$$\int_C \frac{1}{z}\, dz$$

around the unit circle C.

5. (20 points) Write $f(z) = \dfrac{1}{(z-i)(z+1)}$ as $\dfrac{A}{(z-i)} + \dfrac{B}{(z+1)}$ and use the identity

$$\frac{1}{(z+a)} = \frac{1}{z}\frac{1}{1+\frac{a}{z}}$$

to write the Laurent series for $f(z)$ in the unbounded annulus $\{|z| > 1\}$.

6. (a) (20 points) Evaluate

$$\int_C \frac{dz}{z^2 - 2z + 10}$$

where C is the circle of radius 3 about $3i$, traversed counterclockwise.

(b) (20 points) Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 - 2x + 10}$.

7. (20 points) Consider $f(z) = 1/p(z)$, where p is a polynomial of degree $n \geq 2$. Show that the sum of the residues of f at all of its poles must equal 0.

8. (20 points) Calculate $\displaystyle\int_0^{\infty} \frac{dx}{x^4+1}$.

9. (20 points) Explain in detail how $\displaystyle\int_0^{\infty} \frac{dx}{x^5+1}$ could be calculated using complex techniques. *You do not have to carry out the computation!*

10. (20 points) Show that

$$\int_0^{2\pi} \frac{d\theta}{\cos^2\theta + 1} = \pi\sqrt{2}.$$

Show all your work explicitly and carefully.