Stony Brook University MAT 341 Fall 2011
Homework Solutions, Chapter 5 (part 2)

§5.7 # 3 List the five lowest frequencies of vibration of a circular membrane.

SOLUTION. The "angular velocities" ($\omega = 2\pi$ frequency) of the various modes are, as listed on p. 341, $\lambda_{0n}c$ and $\lambda_{mn}c$ for $m, n = 1, 2, 3, \ldots$. Since $\lambda_{0n} = \alpha_{0n}/a$ and $\lambda_{mn} = \alpha_{mn}/a$ it is sufficient to choose the five lowest entries from the table on p. 328 of the first few zeroes of the first few Bessel functions. (Note that the values increase as we read to the right and down!) By inspection these are, with their modes $(m, n)$:

2.405 (0,1)

3.832 (1,1)

5.136 (2,1)

5.520 (0,2)

6.380 (3,1)

§5.7 # 4 Sketch the function $J_0(\lambda_n r)$ for $n = 1, 2, 3$.

SOLUTION. This seems to be the same as §5.5 #2.

§5.7 # 9 The nodal curves of $\phi_{03}(r, \theta) = J_0(\lambda_{03} r)$ are concentric circles. What are their radii (as multiples of $a$)? What are the radii of the circles that are the nodal curves of $\phi_{0n}(r, \theta)$ for general $n$?

SOLUTION. By the construction, $\lambda_{03} = \alpha_{03}/a$, where $\alpha_{03}$ is the third zero of the Bessel function $J_0$, so as to meet the boundary condition $J_0(\lambda_{03} a) = 0$. The function $J_0(\lambda_{03} r)$ will also equal zero at points corresponding to the two earlier zeroes of $J_0$, which are $\alpha_{01}$ and $\alpha_{02}$. Those points will be where $\lambda_{03} r = \alpha_{01}$, i.e. $r = \frac{\alpha_{01}}{\alpha_{03}} a$, and where $\lambda_{03} r = \alpha_{02}$, i.e. $r = \frac{\alpha_{02}}{\alpha_{03}} a$.

In general, the nodal circles of $\phi_{0n}(r, \theta) = J_0(\lambda_{0n} r)$ will correspond to the $(n - 1)$ zeroes of $J_0$ which are smaller that $\alpha_{0n}$, i.e. $\alpha_{01}, \ldots, \alpha_{0,n-1}$. The corresponding radii will be, just as above,

$$r = \frac{\alpha_{01}}{\alpha_{0n}} a, \quad \ldots, \quad r = \frac{\alpha_{0,n-1}}{\alpha_{0n}} a.$$

§5.7 # 10 The nodal curves of $\phi_{mn}(r,\theta)$ are shown in Figure 10 [six-slice pie with 1 interior circle].

a. By examining the figure, determine what values $m$ and $n$ have.

   SOLUTION. We know that $\phi_{mn}(r,\theta)$ has the form $J_m(\lambda_n r)\sin(m\theta)$. Here the function is zero at angles $0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3$ so $m = 3$. The function is also zero for one intermediate value of $r$: we must have $n = 1$, and the nodal circle has radius $r = \frac{\alpha_{31}}{\alpha_{32}}a$ if the radius of the disc is $a$.

b. What is the numerical value of the eigenvalue $\lambda_{32}$ (as a multiple of $a$) for this eigenfunction.

   SOLUTION. From the table on p. 328 we read $\alpha_{32} = 9.761$, so $\lambda_{32} = \frac{1}{a}\alpha_{32} = \frac{1}{a}9.761$.

c. What is the formula for the function $\phi_{mn}(r,\theta)$ whose nodal curves are shown?

   SOLUTION. Assuming that the picture is drawn with the $x$-axis horizontal, the function is $J_3(\lambda_2 r)\sin(3\theta)$.

d. What of the frequency of vibration for the drumhead when it is vibrating in this mode?

   SOLUTION. The corresponding function of $t$ is of the form $\cos(\lambda_{32}ct)$, a function with frequency $\frac{1}{2\pi}\frac{1}{a}\alpha_{32}c$. You would need to know the radius of the drum and the speed of sound in the drumhead to get a numerical answer.