Stony Brook University MAT 341 Fall 2011
Homework Solutions, Chapter 5

§5.3 # 10 Show that the function

$$u_{mn}(x, y, t) = \sin(\mu_m x)\sin(\nu_n y)\cos(\lambda_{mn} ct)$$

where $\mu_m = \frac{m\pi}{a}, \nu_n = \frac{n\pi}{b}$, and $\lambda_{mn}^2 = \mu_m^2 + \nu_n^2$ is a solution of the two-dimensional wave equation on the rectangle $0 < x < a$, $0 < y < b$, with $u = 0$ on the boundary.

SOLUTION. This involves plugging u into the equation $\nabla^2 u = \frac{1}{c^2}\frac{\partial u}{\partial t}$ and checking that $u_{mn} = 0$ if $x = 0$, $x = a$, $y = 0$, or $y = b$.

§5.3 # 11 The places where $u_{mn}(x, y, t) = 0$ for all t are called nodal lines. Describe the nodal lines for

$$(m, n) = (1, 2), (2, 3), (3, 2), (3, 3).$$

SOLUTION. Take $(3, 2)$ for example. The function $u_{32}(x, y, t)$ is zero for all t for points (x, y) where $\sin(\frac{3\pi x}{a})\sin(\frac{2\pi y}{b}) = 0$, i.e. where either one of the sines is zero. In the interval $[0, a]$, $\sin(\frac{3\pi x}{a}) = 0$ when $x = 0, a/3, 2a/3, a$, since then $\frac{3\pi x}{a}$ is an integer multiple of π. Besides the vertical borders, this gives vertical nodal lines $x = a/3, 0 < y < b$ and $x = 2a/3, 0 < y < b$. Similarly the points in the rectangle where $\sin(\frac{2\pi y}{b}) = 0$ satisfy $y = 0, b/2, b$. Besides the horizontal borders, this gives an additional, horizontal, nodal line $y = b/2, 0 < x < a$.

§5.3 # 12 Determine the frequencies of vibration for the functions u_{mn} of Exercise 10. Are there different pairs (m, n) that have the same frequency if $a = b$?

SOLUTION. $\sin(\lambda_{mn} ct)$ has period $2\pi/(\lambda_{mn} c)$ and frequency $\lambda_{mn} c/(2\pi)$ cycles per second (Hertz), if t is measured in seconds. For example u_{32} has frequency $\lambda_{32} c/(2\pi) =$

$$\frac{c\sqrt{(\frac{3\pi}{a})^2 + (\frac{2\pi}{b})^2}}{2\pi} = \frac{c}{2}\sqrt{(\frac{3}{a})^2 + (\frac{2}{b})^2}.$$

When $a = b$, u_{mn} has frequency $\frac{c}{2a}\sqrt{m^2 + n^2}$. Clearly the pairs (m, n) and (n, m) have the same frequency. On the other hand since $50 = 7^2 + 1^2 = 5^2 + 5^2$ the modes $(7, 1)$ and $(5, 5)$ vibrate with the same frequency. See

`http://mathworld.wolfram.com/SumofSquaresFunction.html`

for more information on this problem; of related interest is the "hearing the shape of a drum" problem: see

`http://en.wikipedia.org/wiki/Hearing_the_shape_of_a_drum.`

§5.4 # 5 Suppose the [wave and heat] problems were to be solved in the half-disk $0 < r < a, \quad 0 < \theta < \pi$, with additional conditions

$$v(r, 0, t) = 0, \quad 0 < r < a, \quad 0 < t$$

$$v(r, \pi, t) = 0, \quad 0 < r < a, \quad 0 < t.$$

What eigenvalue problem arises in place of [the full disk analysis]? Solve it.

SOLUTION. The separation of variables $v(r, \theta, t) = R(r)Q(\theta)T(t)$ goes just as for the full disk, and leads as before to $Q'' + \mu^2 Q = 0$, with general solution $Q(\theta) = a \cos \mu\theta + b \sin \mu\theta$. The new boundary conditions become conditions on Q:

$$Q(0) = 0, \quad Q(\pi) = 0.$$

The first condition forces $a = 0$; the second is $b \sin \mu\pi = 0$, which forces $\mu = n$, an integer. So the eigenvalues are the positive integers $1, 2, 3, \ldots$ and the Q-eigenfunctions are $\sin \theta, \sin 2\theta, \sin 3\theta, \ldots$.

§5.5 # 1 Find the values of the parameter λ for which the following problem has a non-zero solution:

$$\frac{1}{r}\frac{d}{dr}(r\frac{d\phi}{dr}) + \lambda^2\phi = 0, \quad 0 < r < a$$

$$\phi(a) = 0, \quad \phi(0) \text{ bounded.}$$

SOLUTION. This is Bessel's Equation with $\mu = 0$. The solution which is bounded at 0 is the Bessel function $J_0(\lambda r)$. The boundary condition $\phi(a) = 0$ means that λa must be one of the zeroes of J_0; the first four are listed on page 328. So the possible values of λ are $2.405/a, 5.520/a, 8.654/a, \ldots$

§5.5 # 2 Sketch the first few eigenfunctions found in exercise 1.

SOLUTION. Use the graph of J_0 on page 329, and suppose a = 5 for the sketch. The graph on [0,5] of the first eigenfunction is obtained by stretching the page 329 graph horizontally to drag the first zero from 2.405 to 5, and then discarding the rest of the graph. For the second eigenfunction, compress the page 329 graph horizontally to drag the second zero from 5.420 to 5, and discard the rest. For the third, compress to drag 8.654 to 5. Etc.

§5.5 # 10 Using the result of exercise 4, solve the eigenvalue problem

$$\frac{1}{r}\frac{d}{dr}(r\frac{d\phi}{dr}) + \lambda^2\phi = 0, \quad 0 < r < a$$

$$\frac{d\phi}{dr}(a) = 0, \quad \phi(0) \text{ bounded.}$$

SOLUTION. The equation as before is Bessel's Equation with $\mu = 0$; the solution which is bounded at 0 is $J_0(\lambda r)$. The possible values of λ are determined by the boundary condition $\frac{d\phi}{dr}(a) = 0$, i.e.

$$\frac{d}{dr}J_0(\lambda r)|_a = 0.$$

From exercise 4 we have

$$\frac{d}{dr}J_0(\lambda r) = -\lambda J_1(\lambda r)$$

so our boundary condition becomes

$$\lambda J_1(\lambda a) = 0.$$

So λ must be 0 or $\frac{1}{a}$ times one of the zeroes of the Bessel function J_1; the first few are listed on p. 328: $\lambda = 3.832/a, 7.016/a, \ldots.$

§5.6 #7 Solve the general problem of determining the temperature in a cylinder with an isulated cylindrical surface [in a disk with an insulated boundary, and no θ-dependence]. The problem is

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v}{\partial r}) = \frac{1}{k}\frac{\partial v}{\partial t}, \quad 0 < r < a, \quad 0 < t$$
$$\frac{\partial v}{\partial r}(a,t) = 0, \quad 0 < t$$
$$v(r,0) = f(r), \quad 0 < r < a.$$

The separation of variables $v(r,t) = \phi(r)T(t)$ goes as in the text and leads to

$$T'' + \lambda^2 T = 0, \quad 0 < t$$
$$(r\phi')' + \lambda^2 r\phi = 0, \quad 0 < r < a.$$

The second equation is Bessel's Equation with $\mu = 0$; the solution bounded at $r = 0$ is $\phi(r) = J_0(\lambda r)$. The eigenvalues λ are determined by the boundary condition which becomes $\phi'(a) = 0$. These are worked out in §5.5 #10 above: $\lambda_0 = 0$, $\lambda_1 = 3.832/a$, $\lambda_2 = 7.016/a, \ldots$. Using superposition, the general solution is

$$v(r,t) = \sum_0^\infty a_n J_0(\lambda_n r)e^{-k\lambda_n^2 t}.$$

The coefficients a_n are determined by the initial condition

$$v(r,0) = f(r) = \sum_0^\infty a_n J_0(\lambda_n r).$$

There is some more work to do: the orthogonality we need is different from that described on p. 333, since these λs come from the zeroes of J_1, not J_0. But the same double-integration-by-parts proof works in this case also:

$$\int_0^a J_0(\lambda_m r)J_0(\lambda_n r)r\ dr = 0, \quad n \neq m$$

and (see §5.6 #8)

$$\int_0^a [J_0(\lambda_n r)]^2 r\ dr = \frac{a^2}{2}[J_0(\lambda_n a)]^2$$

so

$$a_n = \frac{2}{a^2[J_0(\lambda_n a)]^2}\int_0^a f(r)J_0(\lambda_n r)r\ dr.$$