Solve the potential equation in the rectangle $0 < x < a$, $0 < y < b$, subject to the boundary conditions $u(a, y) = 1$, $0 < y < b$, and $u = 0$ on the rest of the boundary.

SOLUTION: The boundary conditions are homogeneous with respect to $y$ so when we separate variables $u(x, y) = X(x)Y(y)$,

$$\nabla^2 u = X''Y + XY'' = 0, \quad \frac{X''}{X} + \frac{Y''}{Y} = 0, \quad \frac{Y''}{Y} = -\frac{X''}{X} = c$$

(with boundary conditions $Y(0) = Y(b) = 0$ and $X(0) = 0$) the constant $c$ should be negative, i.e. $-\lambda^2$. In that case the general solution for $Y$ is

$$Y(y) = a_\lambda \cos(\lambda x) + b_\lambda \sin(\lambda x).$$

The boundary conditions on $Y$ force first $a_\lambda = 0$ and then $\lambda = \frac{n\pi}{b}$: the $n$-th eigenfunction is $\sin \frac{n\pi y}{b}$.

The $n$-th corresponding $X$-equation is $X'' = (\frac{n\pi}{b})^2 X$ with general solution

$$X_n(x) = A_n \cosh \frac{n\pi x}{b} + B_n \sinh \frac{n\pi x}{b}.$$  

The boundary condition $X(0) = 0$ forces $A_n = 0$.

Since these conditions are homogeneous, and the equation is linear, we can use superposition to write

$$u(x, y) = \sum_{1}^{\infty} B_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$$

and use the boundary condition $u(a, y) = 1$ to determine the coefficients $\{B_n\}$. In fact

$$u(a, y) = \sum_{1}^{\infty} B_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b} = 1$$

means that $B_n \sinh \frac{n\pi a}{b}$ is the $n$-th Fourier sine coefficient of the function equal to 1 on $[0, b]$, i.e. the $n$-th Fourier coefficient of the square wave: $\frac{4}{\pi n}$ if $n$ is odd, 0 if $n$ is even. So if $n$ is odd, then

$$B_n = \frac{4}{\pi n} \frac{1}{\sinh \frac{n\pi a}{b}}$$
and $B_n = 0$ if $n$ is even.

§4.3 # 1 Solve the problem consisting of the potential equation on the rectangle $0 < x < a$, $0 < y < b$ with the given boundary conditions. Two of the three are very easy if a polynomial is subtracted from $u$.

(a). $\frac{\partial u}{\partial x}(0, y) = 0; u = 1$ on the rest of the boundary.

SOLUTION. Follow hint, set $v(x, y) = u(x, y) - 1$. Then $\nabla^2 v = \nabla^2 u = 0$ and the boundary conditions become $\frac{\partial v}{\partial x}(0, y) = 0; u = 0$ on the rest of the boundary. The solution to this problem is clearly the constant function $v = 0$, so the solution to the given problem is the constant function $u(x, y) = 1$.

(b). $\frac{\partial u}{\partial x}(0, y) = 0; \frac{\partial u}{\partial x}(a, y) = 0; u(x, 0) = 0; u(x, b) = 1$.

SOLUTION. Follow hint, set $v(x, y) = u(x, y) - \frac{y}{b}$. Then $\nabla^2 v = \nabla^2 u = 0$ and the boundary conditions become $\frac{\partial v}{\partial x}(0, y) = 0; \frac{\partial v}{\partial x}(a, y) = 0; u(x, 0) = 0; u(x, b) = 0$. The solution to this problem is clearly the constant function $v = 0$, so the solution to the given problem is the constant function $u(x, y) = \frac{y}{b}$.

(c). $\frac{\partial u}{\partial x}(x, 0) = 0; u(x, b) = 0; u(0, y) = 1; u(a, y) = 0$.

SOLUTION. The condition $\frac{\partial u}{\partial x}(x, 0) = 0$ is equivalent to $u(x, 0) = C$, a constant. Now the problem splits into two problems: $u = u_1 + u_2$ where

$$\nabla^2 u_1 = 0, \quad u_1(0, y) = u_1(a, y) = 0, \quad u_1(x, 0) = C, u_1(x, b) = 0$$

$$\nabla^2 u_2 = 0, \quad u_2(x, 0) = u_1(x, b) = 0, \quad u_2(0, y) = 1, u_2(a, y) = 0.$$  

These each can be solved by the method of §4.2 # 6. To simplify the calculations switch to $v_1(x, y) = u_1(x, b-y)$ and $v_2(x, y) = u_2(a-x, y)$ and then switch back.

§4.4 # 4 Solve the potential problem in the slot $0 < x < a, 0 < y$, for each of these sets of boundary conditions.

(a) $u(0, y) = 0$, $u(a, y) = 0$, $0 < y$; $u(x, 0) = 1$, $0 < x < a$. 

SOLUTION: This is a $u_1$-type problem (homogeneous $x$-boundary conditions) as on p. 279. So when we separate to get

$$\frac{X''}{X} = -\frac{Y''}{Y} = c$$

we should take $c = \lambda^2$; the $X$-eigenfunctions are then $\sin \frac{n\pi x}{a}$ as usual. The $Y_n$ equation then has general solution $A_n e^{\frac{n\pi y}{a}} + B_n e^{-\frac{n\pi y}{a}}$. The requirement that solutions be bounded as $y \to \infty$ forces $A_n = 0$. Note how the choice of exponential solutions rather than hyperbolic-trigonometric simplifies the calculation. The other choice is also legitimate, but will involve more work. The general solution is then

$$u(x, y) = \sum_{0}^{\infty} B_n \sin \frac{n\pi x}{a} e^{-\frac{n\pi y}{a}}.$$ 

The coefficients $B_n$ are determined by the boundary condition $u(x, 0) = 1, 0 < x < a$:

$$\sum_{0}^{\infty} B_n \sin \frac{n\pi x}{a} = 1.$$ 

This is the sine series for the square wave: $B_n = \frac{4}{n\pi}$ if $n$ is odd, 0 if $n$ is even.

(b.) $u(0, y) = 0, \quad u(a, y) = e^{-y}, \quad 0 < y; u(x, 0) = 0, \quad 0 < x < a.$

SOLUTION. Here the boundary condition in $y$ is homogeneous, so we separate as

$$\frac{X''}{X} = -\frac{Y''}{Y} = c$$

with $c$ positive, e.g. $c = \lambda^2$, and we solve for $Y$ first. The general $Y$ solution is $Y = a \cos \lambda y + b \sin \lambda y$. The boundary condition $u(x, 0) = 0$ forces $a = 0$. The corresponding $X$-equation is $X'' = \lambda^2 X$ with general solution $X = A(\lambda) \cosh \lambda x + B(\lambda) \sinh \lambda x$, and the general $u(x, y)$ solution is

$$u(x, y) = \int_{0}^{\infty} (A(\lambda) \cosh \lambda x + B(\lambda) \sinh \lambda x) \sin \lambda y \, d\lambda.$$ 

The coefficient functions $A(\lambda)$ and $B(\lambda)$ are determined by the boundary conditions $u(0, y) = 0, \quad u(a, y) = e^{-y}$:

$$u(0, y) = \int_{0}^{\infty} A(\lambda) \sin \lambda y \, d\lambda = 0.$$ 

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means that $A(\lambda)$ corresponds to the Fourier sine integral of the zero function, so by uniqueness $A(\lambda) \equiv 0$. Then

$$u(a, y) = \int_0^\infty B(\lambda) \sinh \lambda a \sin \lambda y \, d\lambda = e^{-y}$$

means that $B(\lambda) \sinh \lambda a$ is the Fourier sine integral of $e^{-y}$ so

$$B(\lambda) \sinh \lambda a = \frac{2}{\pi} \int_0^\infty e^{-y} \sin \lambda y \, dy = \frac{2 \lambda}{\pi} \frac{1}{1 + \lambda^2}$$

(integrate by parts or use table of integrals) and

$$B(\lambda) = \frac{2}{\pi} \frac{\lambda}{1 + \lambda^2} \sinh \lambda a.$$

(c.) Similar to (b.), with an easier integral. Make life simpler by setting $v(x, y) = u(a - x, y)$ so $v(0, y) = 0$ and $v(a, y) = f(y)$; switch back to $u$ to finish.

§4.4 # 5 Solve the potential problem in the slot $0 < x < a, 0 < y$, for each of these sets of boundary conditions.

(a.) $\frac{\partial u}{\partial x}(0, y) = 0, u(a, y) = 0, 0 < y; u(x, 0) = 1, 0 < x < a.$

SOLUTION. In this case the $x$-problem is homogeneous, so we separate as

$$X'' = -\frac{Y''}{Y} = -\lambda^2$$

and solve for $X$ first. The general solution is $X(x) = a \cos \lambda x + b \sin \lambda x$; the boundary conditions translate to $X'(0) = 0, X(a) = 0$. The first forces $b = 0$; the second forces $\lambda = \frac{(2n-1)\pi}{2a}$, so the $n$-th $X$-eigenfunction is $\cos \frac{(2n-1)\pi x}{2a}$. The corresponding $Y_n$ equation is

$$Y''_n = \left(\frac{2n - 1}{2a}\right)^2 Y_n$$

with solution $Y_n = a_n \exp\left(\frac{(2n-1)\pi y}{2a}\right) + b_n \exp\left(-\frac{(2n-1)\pi y}{2a}\right)$. Note choice of basis for solutions. The requirement that solutions be bounded as $y \to \infty$ forces $a_n = 0$. The general solution is then

$$u(x, y) = \sum_0^\infty b_n e^{-\frac{(2n-1)\pi y}{2a}} \cos \frac{(2n-1)\pi x}{2a}.$$
where the \( \{b_n\} \) are determined by the boundary condition

\[
u(x, 0) = \sum_{n=0}^{\infty} b_n \cos \frac{(2n-1)\pi x}{2a} = 1.
\]

The functions \( \cos \frac{(2n-1)\pi x}{2a} \) for \( n = 1, 2, 3, \ldots \) are an orthogonal family on \([0, a]\) with

\[
\int_0^a \cos^2 \frac{(2n-1)\pi x}{2a} \, dx = \frac{a}{2},
\]

so

\[
b_n = \frac{2}{a} \int_0^a \cos \frac{(2n-1)\pi x}{2a} \, dx = \frac{2a}{a (2n-1)\pi} \sin \frac{(2n-1)\pi x}{2a} \bigg|_0^a
\]

\[
= \frac{2}{a (2n-1)\pi} \sin \frac{(2n-1)\pi}{2} = \begin{cases} 
\frac{4}{(2n-1)\pi} & n = 1, 3, 5, \ldots \\
-\frac{4}{(2n-1)\pi} & n = 2, 4, 6, \ldots
\end{cases}
\]

(b.) \( \frac{\partial u}{\partial y}(0, y) = 0, u(a, y) = e^{-y}, 0 < y; u(x, 0) = 0, 0 < x < a \). Here the \( y \)-boundary condition is homogeneous, so we separate as

\[
\frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2
\]

and solve for \( Y \) first. The general solution is \( Y(y) = a \cos \lambda y + b \sin \lambda y \); the boundary condition translates to \( Y(0) = 0 \), which forces \( a = 0 \). The \( X \) equation is then \( \frac{X''}{X} = \lambda^2 \), with general solution \( X = A(\lambda) \cosh \lambda x + B(\lambda) \sinh \lambda x \). The general solution for \( u \) is

\[
u(x, y) = \int_0^\infty (A(\lambda) \cosh \lambda x + B(\lambda) \sinh \lambda x) \sin \lambda y \, d\lambda,
\]

where the coefficient functions are determined by the boundary conditions using Fourier integrals. Namely:

\[
\frac{\partial u}{\partial x}(0, y) = \int_0^\infty (\lambda B(\lambda) \sin \lambda y) \, d\lambda = 0
\]

means that \( \lambda B(\lambda) \) is the Fourier sine integral for the zero function; by uniqueness \( \lambda B(\lambda) \equiv 0 \) so \( B(\lambda) \equiv 0 \). Then
\[ u(a, y) = \int_0^\infty A(\lambda) \cosh \lambda a \sin \lambda y \, d\lambda = e^{-y} \]

means that \( A(\lambda) \cosh \lambda a \) is the Fourier sine integral for \( e^{-y} \), i.e.

\[ A(\lambda) \cosh \lambda a = \frac{2}{\pi} \int_0^\infty e^{-y} \sin \lambda y \, dy = \frac{2}{\pi} \frac{\lambda}{1 + \lambda^2} \]

(the same integral as in 4.b.) so

\[ A(\lambda) = \frac{2}{\pi} \frac{\lambda}{1 + \lambda^2} \cosh \lambda a. \]

(c.) \( u(0, y) = 0, u(a, y) = f(y) = \begin{cases} 1, & 0 < y < b \\ 0, & b < y \end{cases}, \ \partial_y u(x, 0) = 0, 0 < x < a. \)

SOLUTION: Here the \( y \)-conditions are homogeneous, so we separate and set \( \frac{Y''}{Y} = -\lambda^2, \ \frac{X''}{X} = \lambda^2 \), and solve for \( Y \) first. The general solution is \( Y = a \cos \lambda y + b \sin \lambda y \); the boundary condition at \( y = 0 \) translates to \( Y'(0) = 0 \), which forces \( b = 0 \), with no condition on \( \lambda \). The corresponding \( X \) equation has general solution \( X(x) = A(\lambda) \cosh \lambda x + B(\lambda) \sinh \lambda x \), leading to

\[ u(x, y) = \int_0^\infty (A(\lambda) \cosh \lambda x + B(\lambda) \sinh \lambda x) \cos \lambda y \, d\lambda \]

where \( A(\lambda) \) and \( B(\lambda) \) are determined by the boundary conditions:

\[ u(0, y) = \int_0^\infty A(\lambda) \cos \lambda y \, d\lambda = 0 \]

gives \( A(\lambda) \) as the cosine integral of the zero function, so \( A(\lambda) \equiv 0 \). Then

\[ u(a, y) = \int_0^\infty B(\lambda) \sinh \lambda a \cos \lambda y \, d\lambda = f(y) \]

gives

\[ B(\lambda) \sinh \lambda a = \frac{2}{\pi} \int_0^\infty f(y) \cos \lambda y \, dy = \frac{2}{\pi} \int_0^b \cos \lambda y \, dy = \frac{2}{\pi \lambda} \sin \lambda b \]

and so

\[ B(\lambda) = \frac{2}{\pi \lambda} \frac{\sin \lambda b}{\sinh \lambda a}. \]
§4.5 #1 Solve the potential equation in the disc \(0 < r < c\) if the boundary condition is \(v(c, \theta) = |\theta|, -\pi < \theta < \pi\).

SOLUTION. As described in §4.5, the potential equation \(\nabla^2 v = 0\) leads, via writing \(v(r, \theta) = R(r)Q(\theta)\), to \(Q'' = -\lambda^2 Q\) with general solution \(Q(\theta) = a \cos \lambda \theta + b \sin \lambda \theta\); since \(Q\) must be periodic of period \(2\pi\) for the function to be well-defined on the disc, \(\lambda\) must be an integer: \(n = 0, 1, 2, 3, ...\) (negative integers don’t give new solutions). The corresponding \(R_n\) must satisfy the Cauchy-Euler equation; the solutions are \(R_n(r) = r^n, R_n(r) = r^{-n}\). The second solution blows up at \(r = 0\) and is not useful. The solution to the problem is then

\[
v(r, \theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n \theta + b_n \sin n \theta) r^n,
\]

where the coefficients \(a_n, b_n\) are determined from the initial conditions by Fourier analysis:

\[
v(c, \theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n \theta + b_n \sin n \theta) c^n = |\theta|.
\]

So \(a_0, a_n c^n\) and \(b_n c^n\) are the coefficients of the Fourier series of \(f(\theta) = |\theta|, -\pi < \theta < \pi\). This \(f\) is an even function, so the sine coefficients are zero, and

\[
a_0 = \frac{1}{\pi} \int_0^\pi \theta \, d\theta = \frac{\pi}{2},
\]

\[
a_n c^n = \frac{2}{\pi} \int_0^\pi \theta \cos n \theta \, d\theta = \begin{cases} \displaystyle \frac{-4}{\pi n^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}
\]

(note that \(|\theta| = \theta\) on \([0, \pi]\)). Finally

\[
v(r, \theta) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{r^n \cos n \theta}{n^2 c^n}.
\]

§4.5 #4 Same as Exercise 1 with boundary condition

\[
v(c, \theta) = f(\theta) = \begin{cases} -1 & -\pi < \theta < 0 \\ 1 & 0 < \theta < \pi \end{cases}.
\]
SOLUTION. Same as Exercise 1, except here $f$ is odd, so the cosine coefficients are zero, and

$$b_n c^n = \frac{2}{\pi} \int_0^\pi \sin n\theta \, d\theta = -\frac{2}{\pi n} \cos n\theta|_0^\pi = \left\{ \begin{array}{ll} \frac{4}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{array} \right..$$

So

$$v(r, \theta) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{nc^n} r^n \sin n\theta.$$  

§4.5 #5 Find the value of the solution at $r = 0$ for the problems of Exercises 1 and 4.

SOLUTION. When $r = 0$ the solution of Exercise 1 gives $v = \frac{\pi}{2}$, and the solution of Exercise 4 gives $v = 0$.  

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