§2.2 #5 Find the steady-state solution of the problem

\[ \frac{\partial}{\partial x} \left( \kappa \frac{\partial u}{\partial x} \right) = \rho \frac{\partial u}{\partial t}, \quad 0 < x < a, \quad 0 < t \]

\[ u(0, t) = T_0, \quad u(a, t) = T_1, \quad 0 < t \]

if the conductivity varies in a linear fashion with \( x \): \( \kappa(x) = \kappa_0 + \beta x \), where \( \kappa_0 \) and \( \beta \) are constants.

SOLUTION: The steady-state solution \( v(x) \) satisfies (see p. 136)

\[ \frac{d}{dx} \left( \kappa(x) \frac{dv}{dx} \right) = 0, \]

so

\[ \kappa(x) \frac{dv}{dx} = C \]

for some constant \( C \). Using \( \kappa(x) = \kappa_0 + \beta x \) this gives

\[ \frac{dv}{dx} = \frac{C}{\kappa_0 + \beta x}. \]

Integrating both sides gives

\[ v(x) = \frac{C}{\beta} \ln(\kappa_0 + \beta x) + D \]

for some other constant \( D \). The boundary conditions for \( u(x, t) \) give boundary conditions for \( v(x) \): \( v(0) = T_0, \ v(a) = T_1 \). These will determine the constants \( C \) and \( D \):

\[ T_0 = \frac{C}{\beta} \ln(\kappa_0) + D \]

\[ T_1 = \frac{C}{\beta} \ln(\kappa_0 + \beta a) + D. \]
Subtracting the two equations gives
\[ T_1 - T_0 = \frac{C}{\beta} [\ln(\kappa_0 + \beta a) - \ln(\kappa_0)] \]
so
\[ C = \frac{\beta(T_1 - T_0)}{\ln(\kappa_0 + \beta a) - \ln(\kappa_0)} \]
and then from the first equation
\[ D = T_0 - \frac{C}{\beta} \ln(\kappa_0) = \frac{(T_1 - T_0) \ln(\kappa_0)}{\ln(\kappa_0 + \beta a) - \ln(\kappa_0)}. \]

§2.2 # 7 Find the steady-state solution of this problem where \( r \) is a constant that represents heat generation.

\[ \frac{\partial^2 u}{\partial x^2} + r = \frac{1}{k} \frac{\partial u}{\partial t}, \quad 0 < x < a, \quad 0 < t \]
\[ u(0, t) = T_0, \quad \frac{\partial u}{\partial x}(a, t) = 0, \quad 0 < t. \]

SOLUTION: The steady-state solution \( v(x) \) must satisfy (see p. 136)

\[ \frac{d^2 v}{dx^2} + r = 0 \]
\[ v(0) = T_0, \quad \frac{dv}{dx}(a) = 0. \]

The differential equation, equivalent to \( \frac{d^2 v}{dx^2} = -r \), has general solution
\[ v(x) = \frac{-r}{2} x^2 + Cx + D \]
where \( C, D \) are the constants of integration. The boundary conditions determine \( C \) and \( D \):
\[ T_0 = v(0) = D \]
\[ 0 = \frac{dv}{dx}(a) = -ra + C \quad \text{so} \quad C = ra. \]
§2.3 # 4 The problem here is locating the definitions of $\phi_1, \phi_2, \phi_3$. They are on page 144 near the bottom. $\phi_n(x) = \sin(\lambda_n x)$. The definition of $\lambda_n = n\pi/a$ is just above on the same page.

§2.3 # 8 Solve the problem

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{k} \frac{\partial w}{\partial t}, \quad 0 < x < a, \quad 0 < t$$

$$w(0, t) = 0, w(a, t) = 0$$

$$w(x, 0) = g(x) = \begin{cases} \frac{2T_0 x}{a} & 0 < x < \frac{a}{2} \\ \frac{2T_0 (a-x)}{a} & \frac{a}{2} < x < a \end{cases}$$

SOLUTION: We follow the analysis on page 143-144 to conclude that the equation and boundary conditions admit product solutions

$$\sin \lambda_n x e^{-k\lambda_n^2 t}$$

with $\lambda_n = n\pi/a$.

Since the equation is linear and the boundary conditions are homogeneous any linear combination of product solutions

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{a} x e^{-k\frac{n^2 \pi^2}{a^2} t}$$

will satisfy the equation and the boundary conditions. We can choose the coefficients $b_n$ to match the initial conditions: at $t = 0$ we need

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{a} x = g(x) = \begin{cases} \frac{2T_0 x}{a} & 0 < x < \frac{a}{2} \\ \frac{2T_0 (a-x)}{a} & \frac{a}{2} < x < a \end{cases}$$

I.e. the $b_n$ are the coefficients of the Fourier sine series of the function on the right, so we know how to calculate them, following Theorem 2 on page 60:

$$b_n = \frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x \, dx.$$
We can save some effort by noting that \( g(x) \) is symmetrical about \( x = \frac{a}{2} \) (check that \( \frac{2T_0(x-a)}{a} = \frac{2T_0(a-(x-a))}{a} \)). Now \( \sin \frac{\pi}{a}x \) is anti-symmetrical about \( x = \frac{a}{2} \) when \( n \) is even, and symmetrical about \( x = \frac{a}{2} \) when \( n \) is odd. This implies that \( b_n = 0 \) if \( n \) is even, and that

\[
b_n = \frac{4}{a} \int_0^\frac{a}{2} g(x) \sin \frac{\pi}{a}x \, dx
\]

when \( n \) is odd. So for odd \( n \):

\[
b_n = \frac{4}{a} \int_0^\frac{a}{2} \frac{2T_0x}{a} \sin \frac{\pi}{a}x \, dx = \frac{8T_0}{a^2} \int_0^\frac{a}{2} x \sin \frac{\pi}{a}x \, dx.
\]

This should be a familiar integration by parts by now:

\[
\int_0^\frac{a}{2} x \sin \frac{\pi}{a}x \, dx = \left. -\frac{a}{n\pi} x \cos \frac{\pi}{a}x \right|_0^{\frac{a}{2}} + \frac{a}{n\pi} \int_0^\frac{a}{2} \cos \frac{\pi}{a}x \, dx
\]

The first term is zero when \( x = 0 \) and also when \( x = \frac{a}{2} \), since it has as a factor the cosine of an odd multiple of \( \frac{\pi}{2} \). Integrating the second term gives

\[
b_n = \frac{8T_0}{a^2} \frac{a^2}{n^2\pi^2} \sin \frac{n\pi}{a}x \left|_0^{\frac{a}{2}} \right. = \frac{8T_0}{n^2\pi^2} \sin \frac{n\pi}{2}.
\]

The sine of an odd multiple \( n \) of \( \frac{\pi}{2} \) is \( 1 \) for \( n = 1, 5, 9, \ldots \) and \( -1 \) for \( n = 3, 7, 11, \ldots \), i.e. it is \( (-1)^j \) if \( n = 2j + 1 \). So we can write our initial condition as

\[
g(x) = \frac{8T_0}{\pi^2} \sum_{n=2j+1,j=0}^\infty (-1)^j \frac{1}{n^2} \sin \frac{n\pi}{a}x,
\]

and the solution to our problem is

\[
w(x, t) = \frac{8T_0}{\pi^2} \sum_{n=2j+1,j=0}^\infty (-1)^j \frac{1}{n^2} \sin \frac{n\pi}{a}x e^{-k\frac{n^2\pi^2}{a^2}t}.
\]

§2.4 #3 This is an insulated bar problem:

\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial w}{\partial t}, \quad 0 < x < a, \quad 0 < t
\]

\[
\frac{\partial u}{\partial x}(0, t) = 0, \quad \frac{\partial u}{\partial x}(a, t) = 0
\]
\[ u(x, 0) = g(x) = \begin{cases} \frac{2T_0 x}{a} & 0 < x < \frac{a}{2} \\ \frac{2T_0 (a-x)}{a} & \frac{a}{2} < x < a \end{cases} \]

**SOLUTION:** The analysis on pages 151 and 152 identifies this as problem which can be solved by a sum of product solutions of the form

\[ a_0, a_n \cos \frac{n\pi}{a} x e^{-k \frac{n^2 \pi^2}{a^2} t}. \]

To satisfy the initial conditions, we need

\[ a_0 + \sum_{n=0}^{\infty} a_n \cos \frac{n\pi}{a} x = g(x) = \begin{cases} \frac{2T_0 x}{a} & 0 < x < \frac{a}{2} \\ \frac{2T_0 (a-x)}{a} & \frac{a}{2} < x < a \end{cases} \]

This involves calculating the Fourier cosine series of \( g(x) \), which we can do following Theorem 2 page 30:

\[ a_0 = \frac{1}{a} \int_0^a g(x) \, dx = \frac{T_0}{2} \]

\[ a_n = \frac{2}{a} \int_0^a g(x) \cos \frac{n\pi}{a} x \, dx. \]

We can save some effort by observing as in §2.3 #8 that \( g(x) \) is symmetric about \( x = \frac{a}{2} \); in this case \( \cos \frac{n\pi}{a} x \) is symmetric about \( x = \frac{a}{2} \) when \( n \) is even, and antisymmetric when \( n \) is odd. So for \( n \) odd, \( a_n = 0 \), and for \( n \) even,

\[ a_n = \frac{4}{a} \int_0^{\frac{a}{2}} \frac{2T_0 x}{a} \cos \frac{n\pi}{a} x \, dx = \frac{8T_0}{a^2} \int_0^{\frac{a}{2}} x \cos \frac{n\pi}{a} x \, dx. \]

This involves the usual kind of integration by parts:

\[ \int_0^{\frac{a}{2}} x \cos \frac{n\pi}{a} x \, dx = \frac{a}{n\pi} x \sin \frac{n\pi}{a} x \bigg|_{0}^{\frac{a}{2}} - \frac{a}{n\pi} \int_0^{\frac{a}{2}} \sin \frac{n\pi}{a} x \, dx. \]
The first term is zero at both endpoints, since $n$ is even. The second term integrates to

$$\frac{a^2}{n^2\pi^2} \cos n\pi x \bigg|_0^\frac{a}{2} = \frac{a^2}{n^2\pi^2} \{\cos n\frac{\pi}{2} - 1\}. $$

If $n$ is even, say $n = 2j$, then $\cos n\frac{\pi}{2} = \cos j\pi$ and is 1 if $j$ is even, $-1$ if $j$ is odd. So our integral is

$$\int_0^\frac{a}{2} x \cos \frac{n\pi x}{a} \, dx = \begin{cases} \frac{a^2}{n^2\pi^2}(-2) & n = 2j, \ j \text{ odd} \\ 0 & n = 2j, \ j \text{ even} \end{cases}$$

and

$$a_n = \begin{cases} \frac{-16T_0}{n^2\pi^2} & n = 2j, \ j \text{ odd} \\ 0 & n = 2j, \ j \text{ even} \end{cases}$$

So

$$g(x) = \frac{T_0}{2} - \frac{16T_0}{\pi^2} \sum_{n=2j, j \text{ odd}}^\infty \frac{1}{n^2} \cos \frac{n\pi x}{a}$$

or

$$g(x) = \frac{T_0}{2} - \frac{4T_0}{\pi^2} \sum_{j \text{ odd}}^\infty \frac{1}{j^2} \cos 2j \frac{\pi x}{a}$$

and the solution to the problem is

$$u(x, t) = \frac{T_0}{2} - \frac{4T_0}{\pi^2} \sum_{j \text{ odd}}^\infty \frac{1}{j^2} \cos 2j \frac{\pi x}{a} e^{-k\frac{4j^2\pi^2}{a^2}t}. $$