Stony Brook University MAT 341 Fall 2011
Homework Solutions, Chapter 2 (corrected typos in §2.3 #8, 10/3, 4:30 PM)

§2.2 #5 Find the steady-state solution of the problem

$$\frac{\partial}{\partial x}(\kappa\frac{\partial u}{\partial x}) = c\rho\frac{\partial u}{\partial t}, \quad 0 < x < a, \quad 0 < t$$

$$u(0,t) = T_0, \quad u(a,t) = T_1, \quad 0 < t$$

if the conductivity varies in a linear fashion with $x$: $\kappa(x) = \kappa_0 + \beta x$, where $\kappa_0$ and $\beta$ are constants.

SOLUTION: The steady-state solution $v(x)$ satisfies (see p. 136)

$$\frac{d}{dx}(\kappa(x)\frac{dv}{dx}) = 0,$$

so

$$\kappa(x)\frac{dv}{dx} = C$$

for some constant $C$. Using $\kappa(x) = \kappa_0 + \beta x$ this gives

$$\frac{dv}{dx} = \frac{C}{\kappa_0 + \beta x}.$$

Integrating both sides gives

$$v(x) = \frac{C}{\beta}\ln(\kappa_0 + \beta x) + D$$

for some other constant $D$. The boundary conditions for $u(x,t)$ give boundary conditions for $v(x)$: $v(0) = T_0,\ v(a) = T_1$. These will determine the constants $C$ and $D$:

$$T_0 = \frac{C}{\beta}\ln(\kappa_0) + D$$

$$T_1 = \frac{C}{\beta}\ln(\kappa_0 + \beta a) + D.$$

Subtracting the two equations gives

$$T_1 - T_0 = \frac{C}{\beta}[\ln(\kappa_0 + \beta a) - \ln(\kappa_0)]$$

so

$$C = \frac{\beta(T_1 - T_0)}{\ln(\kappa_0 + \beta a) - \ln(\kappa_0)}$$

and then from the first equation

$$D = T_0 - \frac{C}{\beta}\ln(\kappa_0) = \frac{(T_1 - T_0)\ln(\kappa_0)}{\ln(\kappa_0 + \beta a) - \ln(\kappa_0)}.$$

§2.2 # 7 Find the steady-state solution of this problem where $r$ is a constant that represents heat generation.

$$\frac{\partial^2 u}{\partial x^2} + r = \frac{1}{k}\frac{\partial u}{\partial t}, \quad 0 < x < a, \quad 0 < t$$

$$u(0,t) = T_0, \quad \frac{\partial u}{\partial x}(a,t) = 0, \quad 0 < t.$$

SOLUTION: The steady-state solution $v(x)$ must satisfy (see p. 136)

$$\frac{d^2 v}{dx^2} + r = 0$$

$$v(0) = T_0, \quad \frac{dv}{dx}(a) = 0.$$

The differential equation, equivalent to $\frac{d^2v}{dx^2} = -r$, has general solution

$$v(x) = \frac{-r}{2}x^2 + Cx + D$$

where $C, D$ are the constants of integration. The boundary conditions determine $C$ and $D$:

$$T_0 = v(0) = D$$

$$0 = \frac{dv}{dx}(a) = -ra + C \quad \text{so } C = ra.$$

§2.3 # 4 The problem here is locating the definitions of $\phi_1, \phi_2, \phi_3$. They are on page 144 near the bottom. $\phi_n(x) = \sin(\lambda_n x)$. The definition of $\lambda_n = n\pi/a$ is just above on the same page.

§2.3 # 8 Solve the problem

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial\ t}, \quad 0 < x < a, \quad 0 < t$$

$$w(0,t) = 0, w(a,t) = 0$$

$$w(x,0) = g(x) = \begin{cases} \frac{2T_0 x}{a} & 0 < x < \frac{a}{2} \\ \frac{2T_0(a-x)}{a} & \frac{a}{2} < x < a \end{cases}$$

SOLUTION: We follow the analysis on page 143-144 to conclude that the equation and boundary conditions admit product solutions

$$\sin \lambda_n x\ e^{-k\lambda_n^2 t}$$

with $\lambda_n = n\frac{\pi}{a}$.

Since the equation is linear and the boundary conditions are homogeneous any linear combination of product solutions

$$\sum_{n=1}^{\infty} b_n \sin n\frac{\pi}{a}x\ e^{-k\frac{n^2\pi^2}{a^2}t}$$

will satisfy the equation and the boundary conditions. We can choose the coefficients $b_n$ to match the initial conditions: at $t = 0$ we need

$$\sum_{n=1}^{\infty} b_n \sin n\frac{\pi}{a}x = g(x) = \begin{cases} \frac{2T_0 x}{a} & 0 < x < \frac{a}{2} \\ \frac{2T_0(a-x)}{a} & \frac{a}{2} < x < a \end{cases}$$

I.e. the $b_n$ are the coefficients of the Fourier sine series of the function on the right, so we know how to calculate them, following Theorem 2 on page 60:

$$b_n = \frac{2}{a}\int_0^a g(x)\sin n\frac{\pi}{a}x\ dx.$$

We can save some effort by noting that $g(x)$ is symmetrical about $x = \frac{a}{2}$ (check that $\frac{2T_0(\frac{a}{2}-x)}{a} = \frac{2T_0(a-(\frac{a}{2}+x))}{a}$). Now $\sin n\frac{\pi}{a}x$ is anti-symmetrical about $x = \frac{a}{2}$ when $n$ is even, and symmetrical about $x = \frac{a}{2}$ when $n$ is odd. This implies that $b_n = 0$ if $n$ is even, and that

$$b_n = \frac{4}{a}\int_0^{\frac{a}{2}} g(x)\sin n\frac{\pi}{a}x\ dx$$

when $n$ is odd. So for odd $n$:

$$b_n = \frac{4}{a}\int_0^{\frac{a}{2}} \frac{2T_0x}{a}\sin n\frac{\pi}{a}x\ dx = \frac{8T_0}{a^2}\int_0^{\frac{a}{2}} x\sin n\frac{\pi}{a}x\ dx.$$

This should be a familiar integration by parts by now:

$$\int_0^{\frac{a}{2}} x\sin n\frac{\pi}{a}x\ dx = \frac{-a}{n\pi}x\cos n\frac{\pi}{a}x|_0^{\frac{a}{2}} + \frac{a}{n\pi}\int_0^{\frac{a}{2}}\cos n\frac{\pi}{a}x\ dx$$

The first term is zero when $x = 0$ and also when $x = \frac{a}{2}$, since it has as a factor the cosine of an odd multiple of $\frac{\pi}{2}$. Integrating the second term gives

$$b_n = \frac{8T_0}{a^2}\frac{a^2}{n^2\pi^2}\sin n\frac{\pi}{a}x|_0^{\frac{a}{2}} = \frac{8T_0}{n^2\pi^2}\sin n\frac{\pi}{2}.$$

The sine of an odd multiple $n$ of $\frac{\pi}{2}$ is 1 for $n = 1, 5, 9, \ldots$ and $-1$ for $n = 3, 7, 11, \ldots$, i.e. it is $(-1)^j$ if $n = 2j+1$. So we can write our initial condition as

$$g(x) = \frac{8T_0}{\pi^2}\sum_{n=2j+1,j=0}^{\infty}(-1)^j\frac{1}{n^2}\sin n\frac{\pi}{a}x,$$

and the solution to our problem is

$$w(x,t) = \frac{8T_0}{\pi^2}\sum_{n=2j+1,j=0}^{\infty}(-1)^j\frac{1}{n^2}\sin n\frac{\pi}{a}x\ e^{-k\frac{n^2\pi^2}{a^2}t}.$$

§2.4 #3 This is an insulated bar problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial\ t},\quad 0 < x < a,\quad 0 < t$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(a,t) = 0$$

$$u(x,0) = g(x) = \begin{cases} \frac{2T_0 x}{a} & 0 < x < \frac{a}{2} \\ \frac{2T_0(a-x)}{a} & \frac{a}{2} < x < a \end{cases}$$

SOLUTION: The analysis on pages 151 and 152 identifies this as problem which can be solved by a sum of product solutions of the form

$$a_0$$

$$a_n \cos n\frac{\pi}{a}x \; e^{-k\frac{n^2\pi^2}{a^2}t}.$$

To satisfy the initial conditions, we need

$$a_0 + \sum_{n=0}^{\infty} a_n \cos n\frac{\pi}{a}x = g(x) = \begin{cases} \frac{2T_0 x}{a} & 0 < x < \frac{a}{2} \\ \frac{2T_0(a-x)}{a} & \frac{a}{2} < x < a \end{cases}$$

This involves calculating the Fourier cosine series of $g(x)$, which we can do following Theorem 2 page 30:

$$a_0 = \frac{1}{a}\int_0^a g(x)\; dx = \frac{T_0}{2}$$

$$a_n = \frac{2}{a}\int_0^a g(x)\cos n\frac{\pi}{a}x\; dx.$$

We can save some effort by observing as in §2.3 #8 that $g(x)$ is symmetric about $x = \frac{a}{2}$; in this case $\cos n\frac{\pi}{a}x$ is symmetric about $x = \frac{a}{2}$ when $n$ is even, and antisymmetric when $n$ is odd. So for $n$ odd, $a_n = 0$, and for $n$ even,

$$a_n = \frac{4}{a}\int_0^{\frac{a}{2}} \frac{2T_0 x}{a}\cos n\frac{\pi}{a}x\; dx = \frac{8T_0}{a^2}\int_0^{\frac{a}{2}} x\cos n\frac{\pi}{a}x\; dx.$$

This involves the usual kind of integration by parts:

$$\int_0^{\frac{a}{2}} x\cos n\frac{\pi}{a}x\; dx = \frac{a}{n\pi}x\sin n\frac{\pi}{a}x|_0^{\frac{a}{2}} - \frac{a}{n\pi}\int_0^{\frac{a}{2}} \sin n\frac{\pi}{a}x\; dx.$$

The first term is zero at both endpoints, since $n$ is even. The second term integrates to

$$\frac{a^2}{n^2\pi^2}\cos n\frac{\pi}{a}x|_0^{\frac{a}{2}} = \frac{a^2}{n^2\pi^2}[\cos n\frac{\pi}{2} - 1].$$

If $n$ is even, say $n = 2j$, then $\cos n\frac{\pi}{2} = \cos j\pi$ and is 1 if $j$ is even, $-1$ if $j$ is odd. So our integral is

$$\int_0^{\frac{a}{2}} x\cos n\frac{\pi}{a}x\;dx = \begin{cases} \frac{a^2}{n^2\pi^2}(-2) & n = 2j, \quad j \text{ odd} \\ 0 & n = 2j, \quad j \text{ even} \end{cases}$$

and

$$a_n = \begin{cases} \frac{-16T_0}{n^2\pi^2} & n = 2j, \quad j \text{ odd} \\ 0 & n = 2j, \quad j \text{ even} \end{cases}$$

So

$$g(x) = \frac{T_0}{2} - \frac{16T_0}{\pi^2}\sum_{n=2j,j \text{ odd}=1}^{\infty}\frac{1}{n^2}\cos n\frac{\pi}{a}x$$

or

$$g(x) = \frac{T_0}{2} - \frac{4T_0}{\pi^2}\sum_{j \text{ odd}=1}^{\infty}\frac{1}{j^2}\cos 2j\frac{\pi}{a}x$$

and the solution to the problem is

$$u(x,t) = \frac{T_0}{2} - \frac{4T_0}{\pi^2}\sum_{j \text{ odd}=1}^{\infty}\frac{1}{j^2}\cos 2j\frac{\pi}{a}x\; e^{-k\frac{4j^2\pi^2}{a^2}t}.$$