Stony Brook University MAT 341 Fall 2011
Homework Solutions, Section 1.2, Problems 1, 7, 11

§1.2 # 1 Find the Fourier series of the following functions, and sketch the graph of their periodic extensions for at least two periods.

a. $f(x) = |x|, \qquad -1 < x < 1$

SOLUTION: $f(x) = |x|$ is an *even* function. Consequently its Fourier series only has cosines, and the integrals are

$$a_0 = \frac{1}{2a}\int_{-a}^{a} f(x)\ dx = \int_0^1 f(x)\ dx$$

$$a_n = \frac{1}{a}\int_{-a}^{a} f(x)\cos n\frac{\pi}{a}x\ dx = 2\int_0^1 f(x)\cos n\pi x\ dx$$

since here $a = 1$, using Theorem 2, p. 60.

Between 0 and 1, $|x| = x$, so

$$a_0 = \int_0^1 x\ dx = \frac{1}{2}x^2|_0^1 = \frac{1}{2}.$$

$$a_n = 2\int_0^1 x\cos n\pi x\ dx.$$

We integrate by parts taking $u = x$ and $dv = \cos n\pi x\ dx$, so $v = \frac{1}{n\pi}\sin n\pi x$ and $du = dx$. Consequently

$$a_n = 2[x\frac{1}{n\pi}\sin n\pi x|_0^1 - \frac{1}{n\pi}\int_0^1 \sin n\pi x\ dx] = -\frac{2}{n\pi}\int_0^1 \sin n\pi x\ dx,$$

since the first term is zero at $x = 0$ and $x = 1$. The anti-derivative of $\sin n\pi x$ is $-\frac{1}{n\pi}\cos n\pi x$, so

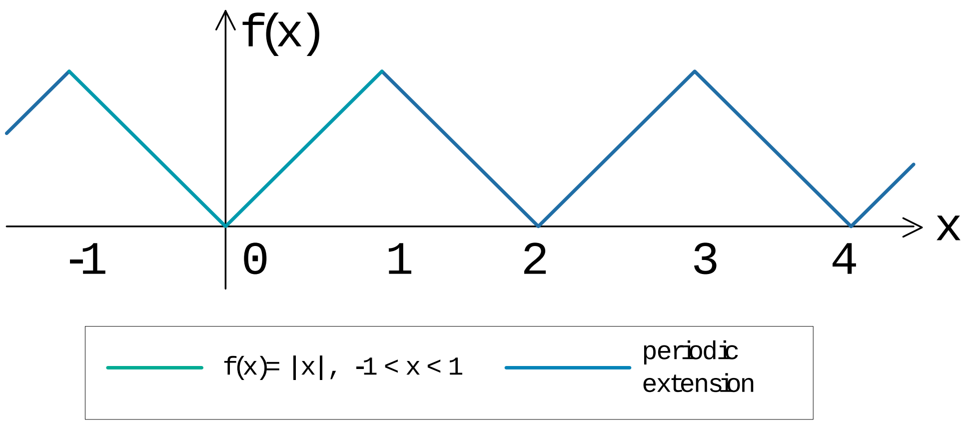
$$a_n = \frac{2}{n^2\pi^2}\cos n\pi x|_0^1.$$

Now $\cos 0 = 1$ and $\cos n\pi = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$, so

$$a_n = \frac{2}{n^2\pi^2}\begin{cases} -1-1 & \text{if } n \text{ is odd} \\ 1-1 & \text{if } n \text{ is even} \end{cases} = \begin{cases} \frac{-4}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

and the Fourier series is

$$f(x) \sim \frac{1}{2} - \frac{4}{\pi^2} \sum_{\substack{n \text{ odd} \\ n=1}}^{\infty} \frac{1}{n^2} \cos n\pi x.$$



b. $f(x) = \begin{cases} -1 & -2 < x < 0 \\ 1 & 0 < x < 2 \end{cases}$

SOLUTION: This is an *odd* function. Consequently only sines will appear in the Fourier series, and the coefficients are

$$b_n = \frac{2}{a} \int_0^a f(x) \sin n\frac{\pi}{a} x \; dx = \int_0^2 f(x) \sin n\frac{\pi}{2} x \; dx$$

since here $a = 2$, and using Theorem 2, p. 60.

Since $f(x) = 1$ on $(0, 2)$, the integral becomes

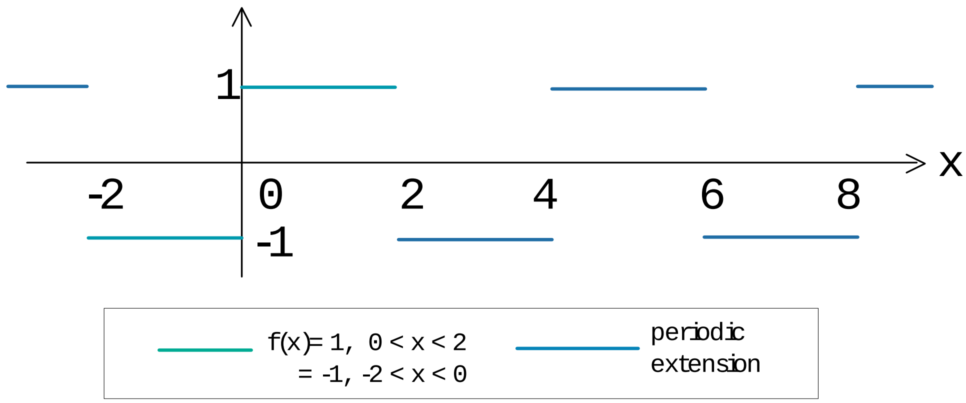
$$b_n = \int_0^2 \sin n\frac{\pi}{2} x \; dx = -\frac{2}{n\pi} \cos n\frac{\pi}{2} x|_0^2.$$

Since $\cos(0) = 1$ and $\cos n\pi = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$ we get

$$b_n = -\frac{2}{n\pi}\left(\begin{Bmatrix} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{Bmatrix} - 1\right) = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

The Fourier series is then

$$f(x) \sim \frac{4}{\pi} \sum_{\substack{n \text{ odd} \\ n=1}}^{\infty} \frac{1}{n} \sin n\frac{\pi}{2}x.$$



§1.2 # 7 Find the Fourier series of the functions:

a. $f(x) = x, \qquad -1 < x < 1$

SOLUTION: This is an odd function, so using Theorem 2 p. 60 only the sines will have non-zero coefficients, and

$$b_n = \frac{2}{a}\int_0^a f(x)\sin n\frac{\pi}{a}x\ dx = 2\int_0^1 x\sin n\pi x\ dx$$

since here $a = 1$ and $f(x) = x$. Integrate by parts, with $u = x$ and $dv = \sin n\pi x\ dx$, so $du = dx$ and $v = -\frac{1}{n\pi}\cos n\pi x$. We get

$$b_n = 2[-\frac{x}{n\pi}\cos n\pi x|_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi x\ dx].$$

In this case the integral is zero since $\sin n\pi x$ equals zero when $x = 1$ and when $x = 0$. Also $\frac{x}{n\pi}\cos n\pi x$ is zero at $x = 0$.
So

$$b_n = \frac{-2}{n\pi}\cos n\pi = \frac{-2}{n\pi}\cdot\begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even}\end{cases} = \begin{cases} \frac{2}{n\pi} & \text{if } n \text{ is odd} \\ \frac{-2}{n\pi} & \text{if } n \text{ is even}\end{cases}$$

and the Fourier series is

$$f(x) \sim \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\sin n\pi x.$$

b. $f(x) = 1, \qquad -2 < x < 2.$

SOLUTION: This function is constant, so it is even, but we can calculate the Fourier coefficients directly, and they do not depend on a:

$$a_0 = \frac{1}{2a}\int_{-a}^{a} 1 \cdot dx = 1$$

$$a_n = \frac{1}{a}\int_{-a}^{a}\cos n\frac{\pi}{a}x\ dx = \frac{1}{a}\frac{a}{n\pi}\sin n\frac{\pi}{a}x|_{-a}^{a} = 0$$

since $\sin n\frac{\pi}{a}a = \sin n\frac{\pi}{a}(-a) = 0$.

$$b_n = \frac{1}{a}\int_{-a}^{a}\sin n\frac{\pi}{a}x\ dx = \frac{1}{a}\frac{-a}{n\pi}\cos n\frac{\pi}{a}x|_{-a}^{a} = 0$$

since $\cos n\frac{\pi}{a}a = \cos n\frac{\pi}{a}(-a)$. So the Fourier series is just $f(x) \sim 1$.

c. $f(x) = \begin{cases} x & \frac{-1}{2} < x < \frac{1}{2} \\ & \\ 1 - x & \frac{1}{2} < x < \frac{3}{2} \end{cases}$

SOLUTION: Look at the graph of f. If we shift it by $\frac{1}{2}$ to the left it becomes the even function $g(x) = f(x + \frac{1}{2})$.

$$g(x) = f(x + \frac{1}{2}) = \begin{cases} x + \frac{1}{2} & \frac{-1}{2} < x + \frac{1}{2} < \frac{1}{2} \\ 1 - (x + \frac{1}{2}) & \frac{1}{2} < x + \frac{1}{2} < \frac{3}{2} \end{cases}$$

or

$$g(x) = \begin{cases} x + \frac{1}{2} & -1 < x < 0 \\ \frac{1}{2} - x & 0 < x < 1 \end{cases}$$

We first calculate the Fourier series for g. It is an even function with $a = 1$, so (using Theorem 2 on p. 60) the b_n are zero and

$$a_0 = \int_0^1 g(x)\ dx = \int_0^1 (\frac{1}{2} - x)\ dx = 0$$

$$a_n = 2\int_0^1 g(x)\cos n\pi x\ dx = 2[\frac{1}{2}\int_0^1 \cos n\pi x\ dx - \int_0^1 x\cos n\pi x\ dx]$$

As we have calculated in an earlier problem, $\int_0^1 \cos n\pi x\ dx = 0$.
Also in problem 1 we calculated $2\int_0^1 x\cos n\pi x\, dx = \begin{cases} \frac{-4}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$
So the Fourier coefficients for $g(x)$ are

$$a_n = -2\int_0^1 x\cos n\pi x\ dx = \begin{cases} \frac{4}{n^2\pi^2} & \text{if } n \text{ is odd} \\ \\ 0 & \text{if } n \text{ is even} \end{cases}$$

and the Fourier series is

$$g(x) \sim \frac{4}{\pi^2} \sum_{\substack{n \text{ odd} \\ n=1}}^{\infty} \frac{1}{n^2} \cos n\pi x.$$

Now we use this series to get the Fourier series for f:

$$f(x) = g(x - \frac{1}{2}) \sim \frac{4}{\pi^2} \sum_{\substack{n \text{ odd} \\ n=1}}^{\infty} \frac{1}{n^2} \cos n\pi(x - \frac{1}{2}) = \frac{4}{\pi^2} \sum_{\substack{n \text{ odd} \\ n=1}}^{\infty} \frac{1}{n^2} \cos(n\pi x - \frac{n\pi}{2}).$$

If n is odd $\cos(y - \frac{n\pi}{2}) = \pm \sin y$, plus if $n = 1, 5, 9, ...$, minus if $n = 3, 7, 11, ...$ So:

$$f(x) \sim \frac{4}{\pi^2}(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - \frac{1}{49}\sin 7x + \text{etc.})$$

or

$$f(x) \sim \frac{4}{\pi^2}\sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)^2}\sin(2n+1)x.$$

§1.2 # 11 Find the Fourier sine and cosine series of the functions:

a. $f(x) = 1, \qquad 0 < x < a$

SOLUTION: **Cosine series**. The even extension of f is the constant function $f(x) = 1$ on $(-a, a)$. As calculated in problem 7b, the only nonzero coefficient is $a_0 = 1$. The cosine series is $f(x) \sim 1$.

Sine series. The odd extension of f is the "square wave"

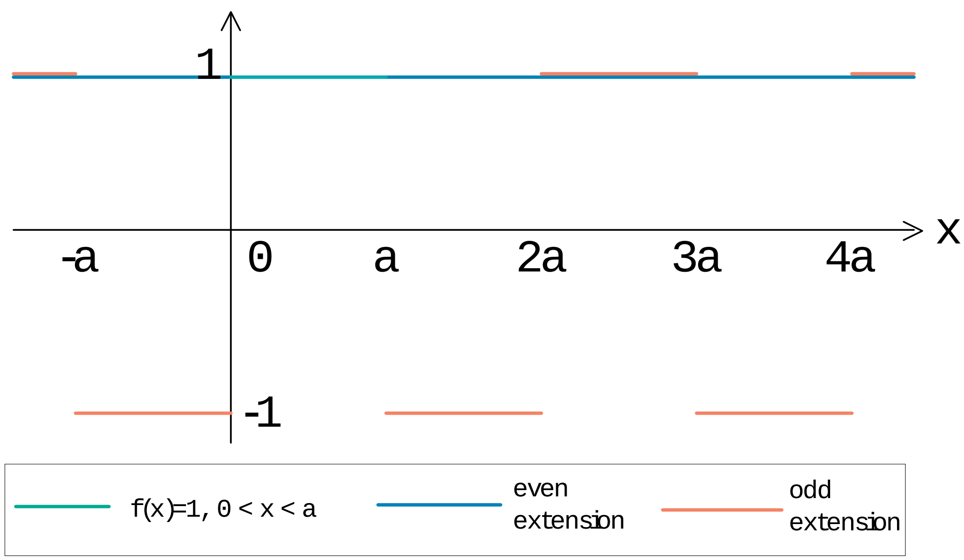
$$f(x) = \begin{cases} -1 & -a < x < 0 \\ 1 & 0 < x < a \end{cases}$$

For an odd function (Theorem 2 p.60) the $a_n = 0$ and $b_n = \frac{2}{a}\int_0^a \sin n\frac{\pi}{a}x\ dx$ since $f(x) = 1$ on that interval. As in problem 7b, this integral is

$$b_n = \frac{2}{a}\frac{-a}{n\pi}\cos n\frac{\pi}{a}x|_0^a = \frac{-2}{n\pi}\begin{cases} -2 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

and the Fourier sine series is

$$f(x) \sim \frac{4}{\pi}\sum_{\substack{n \text{ odd} \\ n=1}}^{\infty} \frac{1}{n}\sin n\frac{\pi}{a}x.$$



b. $f(x) = x, \qquad 0 < x < a$

SOLUTION: **Cosine series**. The even extension of f is

$$f(x) = \begin{cases} -x & -a < x < 0 \\ x & 0 < x < a \end{cases} = |x|, \quad -a < x < a$$

Using Theorem 2 on p.60 as usual, the b_n coefficients are zero,

$$a_0 = \frac{1}{a}\int_0^a x\ dx = \frac{a}{2}$$

$$a_n = \frac{2}{a}\int_0^a x\cos n\frac{\pi}{a}x\ dx.$$

The a_n integral is done by parts as usual, with $u = x$ and $dv = \cos n\frac{\pi}{a}x\ dx$, so $v = \frac{a}{n\pi}\sin n\frac{\pi}{a}x$, $du = dx$ and

$$a_n = x\ \frac{a}{n\pi}\sin n\frac{\pi}{a}x|_0^a - \frac{a}{n\pi}\int_0^a \sin n\frac{\pi}{a}x\ dx = \frac{a^2}{n^2\pi^2}\cos n\frac{\pi}{a}x|_0^a$$

since the first term is zero at both ends, and
$a_n = \frac{a^2}{n^2\pi^2}(\cos(n\pi) - 1) = -2\frac{a^2}{n^2\pi^2}$ if n is odd, and 0 if n is even.

The Fourier cosine series is:

$$f(x) \sim \frac{a}{2} - \frac{2a^2}{\pi^2}\sum_{\substack{n \text{ odd} \\ n=1}}^{\infty} \frac{1}{n^2}\cos n\frac{\pi}{a}x.$$

Sine series. Since $f(x) = x$ is itself an odd function, the function $f(x) = x, \quad -a < x < a$ is the odd extension of $f(x) = x, \quad 0 < x < a$. The Fourier series of this function has no cosine terms, and the sine coefficients are given (see Theorem 2 page 60) by

$$b_n = \frac{2}{a}\int_0^a x\sin n\frac{\pi}{a}x\ dx.$$

We have calculated this before when $a = 1$, in problem 7a. Here again we integrate by parts, with $u = x$ and $dv = \sin n\frac{\pi}{a}x\ dx$, so $v = \frac{-a}{n\pi}\cos n\frac{\pi}{a}x$ and $du = dx$. This gives

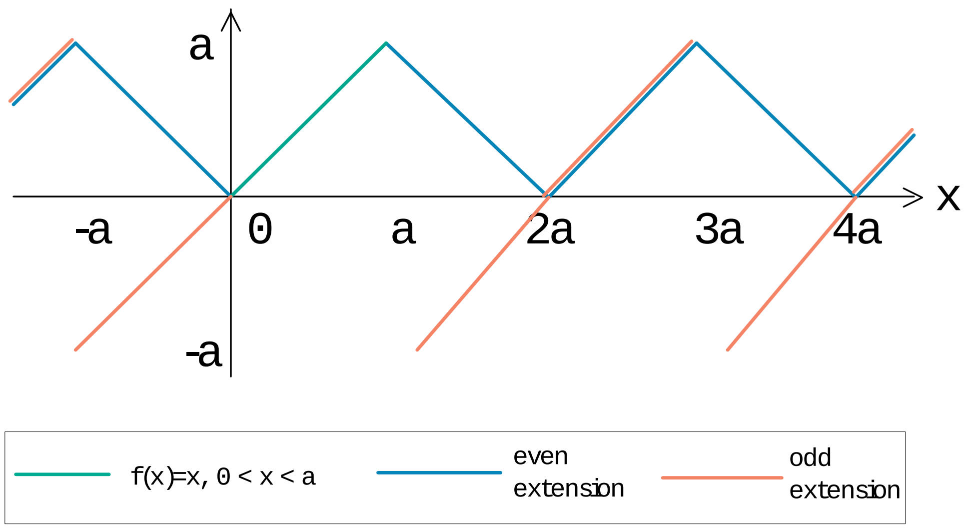
$$b_n = \frac{2}{a}[\frac{-a}{n\pi}x\cos n\frac{\pi}{a}x|_0^a + \frac{a}{n\pi}\int_0^a \cos n\frac{\pi}{a}x\ dx]$$

The integral gives zero since $\sin n\frac{\pi}{a}x$ is 0 when $x = 0$ and when $x = a$. Also $x\cos n\frac{\pi}{a}x$ is 0 when $x = 0$, so what is left is $\frac{-2a}{n\pi}\cos n\pi$, which is $\frac{2a}{n\pi}$ if n is odd, and $\frac{-2a}{n\pi}$ if n is even.

The Fourier sine series is then

$$f(x) \sim \frac{2a}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \sin n \frac{\pi}{a} x.$$

c. $f(x) = \sin x, \quad 0 < x < 1.$



SOLUTION: **Cosine series**. The even extension of f is

$$f(x) = \begin{cases} -\sin x & -1 < x < 0 \\ \sin x & 0 < x < 1 \end{cases}$$

(or $f(x) = |\sin x|, \quad -1 < x < 1$). Using Theorem 2 on page 60, the sine coefficients are all zero, and (here $a = 1$)

$$a_0 = \int_0^1 \sin x \; dx = 1 - \cos 1 = 0.4596...$$

$$a_n = 2 \int_0^1 \sin x \cos n\pi x \; dx.$$

Using the identity $\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$ we can rewrite the integral as

$$a_n = \int_0^1 \sin(1 + n\pi)x \; dx + \int_0^1 \sin(1 - n\pi)x \; dx$$

$$a_n = \frac{-1}{1+n\pi}\cos(1+n\pi)x|_0^1 + \frac{-1}{1-n\pi}\cos(1-n\pi)x|_0^1$$

$$a_n = \frac{1}{1+n\pi}(1-\cos(1+n\pi)) + \frac{1}{1-n\pi}(1-\cos(1-n\pi))$$

so $a_1 = -.3473...$, $a_2 = -.0238...$, $a_3 = -.0350...$, etc.

Sine series. The odd extension of f is the odd function $f(x) = \sin x, \quad -1 < x < 1$. Using Theorem 2 on page 60, all the cosine coefficients are zero, and the sine coefficients are (here $a = 1$)

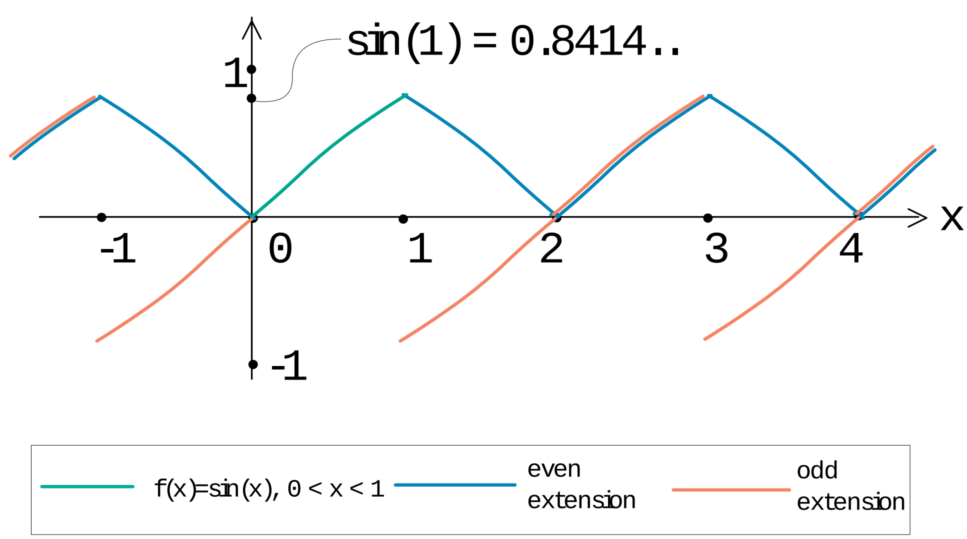
$$b_n = 2\int_0^1 \sin x \sin n\pi x \; dx.$$

Here use the trigonometric identity $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ which gives

$$b_n = \int_0^1 \cos(1-n\pi)x \; dx - \int_0^1 \cos(1+n\pi)x \; dx$$

$$b_n = \frac{1}{1-n\pi}\sin(1-n\pi) - \frac{1}{1+n\pi}\sin(1+n\pi)$$

So $b_1 = .5960...$, $b_2 = -.2748...$, $b_3 = .1805...$, etc.



d. $f(x) = \sin x \quad 0 < x < \pi$.

SOLUTION. **Cosine series**. As in part c., the even extension is $f(x) = |\sin x|, \quad -\pi < x < \pi$; the sine coefficients are all zero;

$$a_0 = \frac{1}{\pi}\int_0^\pi \sin x \, dx = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^\pi \sin x \cos nx \, dx.$$

Notice that orthogonality does not apply since we are only integrating over half a period! Using the trigonometric identity $\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$ as before,

$$a_n = \frac{1}{\pi}\int_0^\pi \sin(1+n)x \, dx + \frac{1}{\pi}\int_0^\pi \sin(1-n)x \, dx$$

$$a_n = \frac{1}{\pi}\frac{-1}{1+n}\cos(1+n)x|_0^\pi + \frac{1}{\pi}\frac{-1}{1-n}\cos(1-n)x|_0^\pi$$

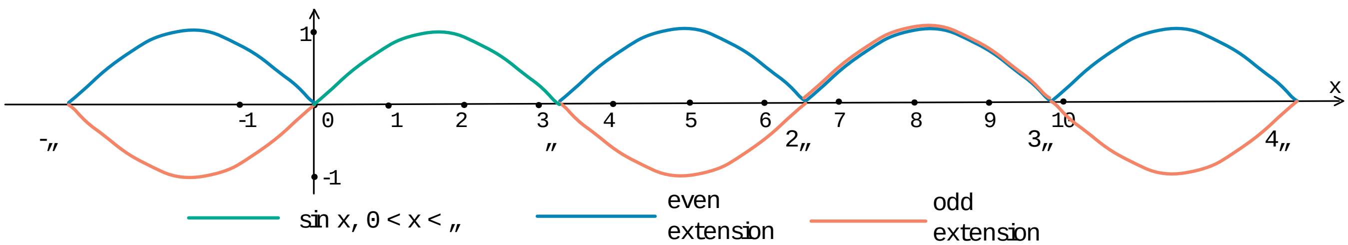
When n is odd, $\cos(1+n)\pi = \cos(1-n)\pi = \cos 0$ so both of the terms give zero. When n is even, $\cos(1+n)\pi = \cos(1-n)\pi = -1$ so

$$a_n = \frac{1}{\pi}(\frac{2}{1+n} + \frac{2}{1-n}) = \frac{4}{\pi} \cdot \frac{1}{1-n^2}$$

and the Fourier cosine series is

$$f(x) \sim \frac{2}{\pi} + \frac{4}{\pi}\sum_{\substack{n \text{ even} \\ n=2}}^{\infty} \frac{1}{1-n^2}\cos nx.$$

Sine series. This function is its own Fourier sine series: $b_1 = 1$ and all the other coeficients are zero.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