My NAME is:

Problem	1	2	3	4	5	6	7	Total
Score								

MAT 341 Applied Real Analysis Final Exam

December 13, 2011 Total score = 140. Each problem is worth 20 points.

This test is open book: Powers "Boundary Value Problems" may be consulted. No other references or notes may be used. Students may use graphing calculators like TI-83, 84, 85, 86; but they may NOT use calculators with Computer Algebra Systems, like TI-89. SHOW ALL YOUR WORK! When using Powers or your calculator be sure to report it, e.g. "from calculator," "from Powers page x."

1. Calculate the Fourier series of the function defined on $[0, 2\pi]$ by

$$f(x) = \begin{cases} 0 & 0 < x < \pi/2 \\ 1 & \pi/2 < x < 3\pi/2 \\ 0 & 3\pi/2 < x < 2\pi \end{cases}$$

and extended to a periodic function of period 2π.

2. A heat-conducting bar of length 10 cm is insulated along its length. Locate points on the bar by x, $0 \leq x \leq 10$. The bar is also insulated at the $(x = 0)$ end. The bar is initially at temperature 1.

   Solve the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad 0 < t$$

$$u(x,0) = 1, \qquad 0 < x < 10$$

   in the bar under the two following different additional boundary conditions.

   a. Starting at $t = 0$ the $(x = 10)$ end is held at temperature 2.

   b. The $(x = 10)$ end of the bar is also insulated.

3. The wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad 0 < x < 100 \qquad 0 < t$$

with boundary conditions

$$u(0,t) = u(100,t) = 0 \qquad 0 < t$$

governs the vibrations of a wire of length 100 cm. At $t = 0$ the wire, initially at rest, is struck so that the points in the interval $[25, 75]$ acquire upward velocity of 5 cm/s. Solve the wave equation with the corresponding initial conditions, i.e.

$$u(x,0) = 0, \quad 0 < x < 100$$

$$\frac{\partial u}{\partial t}(x,0) = f(x) = \begin{cases} 0 & 0 < x < 25 \\ 5 & 25 < x < 75 \\ 0 & 75 < x < 100. \end{cases} \quad .$$

4. Solve the potential equation $\nabla^2 u = 0$ in the infinite strip $0 \leq x \leq a, \quad 0 \leq y$ with boundary conditions

$$u(x,0) = 1, \quad 0 < x < a$$

$$u(0,y) = u(a,y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & 1 < y \end{cases} .$$

5. Solve the potential equation $\nabla^2 v = 0$ in the disk $0 \le r \le c$, $0 \le \theta < 2\pi$ with boundary conditions $v(c,\theta) = f(\theta) = \begin{cases} 1 & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \end{cases}$.

6. An odd, periodic function $f(x)$ of period 2 has Fourier series

$$f(x) \sim \sum_{n \text{ odd } =1}^{\infty} \frac{8}{n^3\pi^3} \sin \frac{n\pi x}{2}.$$

Prove that f is differentiable with a continuous first derivative, but that f does not have a continuous second derivative.

7. (Context: a is a positive number; λ_m is $\frac{1}{a}$ times the m-th zero of the Bessel function $J_0(x)$).

   The eigenfunctions $\phi_m(r) = J_0(\lambda_m r)$ satisfy

$$(r\phi')' + \lambda_m^2 r\phi = 0$$

$$\phi(a) = 0.$$

   Prove carefully that if $m \neq n$ then

$$\int_0^a r\phi_m(r)\phi_n(r)\ dr = 0.$$

End of Examination.