MAT 320 Fall 2009 Review for Midterm 2

- Theorem: be able to apply
- *Theorem*: and know what goes into the proof
- **Theorem**: and be able to prove.

§3.1 Know definition of limit of a sequence and be able to prove **Uniqueness of limits**; this is essential. Understand the examples in this section.

§3.2 **A convergent sequence is bounded**. **Sum, Product of convergent sequences are convergent**. *If* $(x_n) \to x$ *and* $(y_n) \to y \neq 0$ *then* $(x_n/y_n) \to x/y$. Understand Tails of sequences and **A sequence converges if any one of its m-tails converges.**

§3.3 **Monotone Convergence Theorem**. The least upper bound property is crucial. Understand Example 3.3.3(b).

§3.4 Theorem 3.4.4. **Monotone Subsequence Theorem, Bolzano- Weierstrass Theorem (first proof)**.

§3.5 Know definition of Cauchy sequence. **Cauchy Convergence Criterion**. Proof uses **Lemma**: a Cauchy sequence is bounded; then Bolzano-Weierstrass to produce a candidate limit; then additional $\epsilon - \delta$ argument to show that limit works.

Know definition of contractive sequence. **A contractive sequence is a Cauchy sequence**. Proof is straightforward once you use $a^k + a^{k+1} + \cdots + a^{k+\ell} = a^k(1 - a^{\ell+1})/(1 - a)$.

§3.6 Know definition of properly divergent sequence.

§4.1 Know definition of cluster point and of limit of a function at a cluster point. **Theorem 4.1.5** (uniqueness of limit): basic and paradigmatic $\epsilon - \delta$ argument. *Sequential Criterion for Limits (Theorem 4.1.8)*. Divergence Criteria (4.1.9).

§4.2 **Theorem 4.2.2** (f has a limit at c implies f is bounded on a neighborhood of c). Nice $\epsilon - \delta$ argument. *Theorem 4.2.3* on limits of sums, products, quotients. *Theorem 4.2.6* on $\leq$-inequalities persisting to limits.

Theorem 4.2.9 ($\lim_{x\to c} f(x) > 0$ implies that c has a δ-neighborhood on which $f(x) > 0$): useful theorem and illustrative proof.

§4.3 Not necessary to review for test. Check exercises.

§5.1 Know definition of "f continuous at c" in $\epsilon - \delta$ terms. Understand Remark after Theorem 5.1.2. **Sequential Criterion for Continuity**. Know 5.1.6 Examples (g) and (h).

§5.2 *Theorem 5.2.2* on sums, products and quotients of continuous functions. **Theorem 5.2.6** (if f continuous at c and g continuous at $f(c)$ then $g \circ f$ continuous at c): $\epsilon - \delta - \gamma$ argument.

§5.3 has three important theorems. **Theorem 5.3.2** (f continuous on $[a, b]$ implies f bounded): proof by contradiction. Use $\mathbf{N}$ to construct a sequence, use Bolzano-Weierstrass to find a point where f is not continuous. **Maximum Theorem 5.3.4** (f continuous on $[a, b]$ has "a maximum": a point where it takes on its maximum value): use 5.3.2, the least upper bound axiom and $\mathbf{N}$ to define a sequence, and Bolzano-Weierstrass to extract a convergent subsequence; this identifies a candidate maximum; prove this point works. Same for minimum. **"Location of Roots" Theorem** (f continuous on $[a, b]$ and $f(a) < 0 < f(b)$ implies $\exists c \in (a, b)$ with $f(c) = 0$): uses a bisection argument and the Nested Intervals Property. **Intermediate Value Theorem** is direct consequence. A consequence of these theorems is *Theorem 5.3.9*: if f is continuous on $[a, b]$ then $f([a, b])$ is another closed interval.