MAT 320 Fall 2009 Review for Final

- Theorem: be able to apply
- *Theorem*: and know what goes into the proof
- **Theorem**: and be able to prove.

§5.4 Know definition of uniform continuity and be able to show, for example, that $f(x) = 1/x$ on $(0, 1]$, which is continuous, is not uniformly continuous (this is discussed on pp. 136-137). Be familiar with the logical manipulations to get Nonuniformity criteria 5.4.2 (ii) and (iii). **Uniform Continuity Theorem** (f continuous on $[a, b]$ is uniformly continuous): by contradiction using (iii) and Bolzano- Weierstrass to locate a point at which you can show f is not continuous.

Know definition of Lipschitz function. *A Lipschitz function is uniformly continuous.*

Theorem 5.4.7 (a uniformly continuous function takes Cauchy sequences to Cauchy sequences): nice combination of Cauchy criterion with $\epsilon-\delta$ definition of uniform continuity. *Continuous Extension Theorem 5.4.8* is a consequence.

§5.6 Here we will consider functions defined on an interval I (without specifying which if any endpoints are included). Know distinction betweem "increasing" and "strictly increasing," etc. and also "monotone" and "strictly monotone." For f increasing, understand the definition of the jump $j_f(c)$ of f at an interior point c of I (it's $\lim_{x\to c+} f(x) - \lim_{x\to c-} f(x)$) and the definitions of jumps at endpoints. *Theorem 5.6.3* (An increasing f is continuous on I iff $j_f(c) = 0$ for every $c \in I$). And similarly for decreasing. **Theorem 5.6.4** (a monotonic function on an interval (a, b) has at most a countable number of points of discontinuity): at most 1 with jump $\geq (b-a)$, at most 2 with jump $\geq (b - a)/2$, etc., using 5.6.3. **Continuous Inverse Theorem 5.6.5** (a strictly monotone, continuous f defined on an interval I has a continuous inverse g): first g exists because f strictly monotonic; g is also (strictly) monotonic; a discontinuity of g would be a jump; this would force I to be missing a point.

§6.1 Here again f is defined on an interval I. Know the definition of the derivative of f at $c \in I$. **Theorem 6.1.2** (f has a derivative at c implies f

continuous at c): directly from the definition, show $\lim_{x\to c}(f(x) - f(c)) = 0$. *Theorem 6.1.3 - Differentiation Rules* - pay attention to the quotient. **Carathéodory's Theorem 6.1.5** (very useful in getting rid of troublesome denominators): proof is straightforward. **Chain Rule 6.1.6** -use Carathéodory. *Derivative of Inverse* - note requirement that $f'(c) \neq 0$; use Carathéodory.

§6.2 **Interior Extremum Theorem 6.2.1** (if c is an interior extremum of f, then if $f'(c)$ exists, it is 0): straightforward proof by contradiction, using definition of derivative. **Rolle's Theorem 6.2.3 and Mean Value Theorem 6.2.4** both for f continuous on $[a, b]$ and differentiable on (a, b). RT: if $f(a) = f(b)$ then there exists $c \in (a, b)$ with $f'(c) = 0$. Use continuity and Maximum Theorem to find an extremum; show it must be interior; apply 6.2.1. MVT: there exists $c \in (a, b)$ with $f'(c) = (f(b)-f(a))/(b-a)$. Cook up a function expressing the difference between f and the straight-line function from $(a, f(a))$ to $(b, f(b))$, and apply Rolle's Theorem. *Theorems 6.2.5 and 6.2.7* (with same hypotheses: $f'(x) = 0$ for all $a < x < b$ iff f is constant; $f'(x) \geq 0$ for all $a < x < b$ iff f is increasing; $f'(x) \leq 0$ for all $a < x < b$ iff f is decreasing). Directly from MVT and definition of derivative. Note remark on p. 171 about $f(x) = x^3$, etc. *Darboux's Theorem 6.2.12* (f differentiable on $[a, b]$ implies that f' takes on any value k between $f'(a)$ and $f'(b)$) follows from *Lemma 6.2.11* (straightforward from definition of derivative) and the interior extremum theorem applied to $g(x) = kx - f(x)$.

§6.4 Know the definition of the nth Taylor Polynomial $P_n(x)$ approximating a function f at a point x_0. *Taylor's Theorem 6.4.1* ($f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ for some c between x_0 and x): understand that it is proved by applying Rolle's Theorem to an appropriately cooked up auxiliary function. *Newton's Method 6.4.7*: understand how it works and why it gives "quadratic" convergence.

§7.1 Understand the parallelism between the definition of "f is Riemann integrable on $[a, b]$ with integral L" and, for example, "the sequence (a_n) is convergent with limit L." Basic: **Theorem 7.1.2** The integral is unique. Understand examples (c) and (d) on p.198, and understand the elementary **Theorem 7.1.4**. Also *Theorem 7.1.5*, and review Example 7.1.6 (Thomae's function on $[0, 1]$ is in $\mathcal{R}([0, 1])$.

§7.2 *Theorem 7.2.1* (Cauchy Criterion) important because it gives a definition of "f integrable on $[a, b]$" that does not involve the value of $\int_a^b f$. **Theorem**

7.2.3 - "Squeeze Theorem" used in proof of **Theorem 7.2.6**: If f is continuous on $[a, b]$ then $f \in \mathcal{R}([a, b])$.