Solutions

MAT 319/320 MIDTERM 1

Special instructions: *Calculators allowed. Books and records are allowed.*

ID: Name (printed):

Problem 1 : Prove that $\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)} \le \frac{1}{\sqrt{2n+1}}$, for any $n \in \mathbb{N}$.

Proof: By Induction.

① $n=1$ $\quad \frac{1}{2} = \frac{1}{\sqrt{4}} \le \frac{1}{\sqrt{3}}$ (since $\sqrt{3} \le \sqrt{4}$)

② Suppose for $n=k$

$$\frac{1.3.\cdots.(2k-1)}{2.4.\cdots(2k)} \le \frac{1}{\sqrt{2k+1}}$$

③ $n=k+1$

$$\frac{1.3.\cdots.(2k-1).(2k+1)}{2.4.\cdots.(2k).(2k+2)} = \frac{1.3\cdots.(2k-1)}{2.4\cdots.(2k)} \cdot \frac{2k+1}{2k+2} \le$$

$$\le \frac{1}{\sqrt{2k+1}} \cdot \frac{\sqrt{(2k+1)^2}}{\sqrt{(2k+2)^2}} = \frac{\sqrt{2k+1}}{\sqrt{(2k+2)^2}} \le \frac{\sqrt{2k+1}}{\sqrt{(2k+2)^2-1}}$$

$$= \frac{\sqrt{2k+1}}{\sqrt{[(2k+2)-1][(2k+2)+1]}} = \frac{\cancel{\sqrt{2k+1}}}{\cancel{\sqrt{(2k+1)}}\cdot\sqrt{2k+3}} = \frac{1}{\sqrt{2k+3}}$$

$\square$

By ①, ②, ③ $\Rightarrow \frac{1.3.\cdots.(2n-1)}{2.4.\cdots.(2n)} \le \frac{1}{\sqrt{2n+1}}$ $\forall n \in \mathbb{N}$

Problem 2 : Let $f : A \mapsto B$, $F_i \subset B$ for any $i = 1, \dots n$. Prove that

$$f^{-1}(\bigcup_{i=1}^{n} F_i) = \bigcup_{i=1}^{n} f^{-1}(F_i)$$

1 way: Can prove by Induction as in Homework.

2 way: 1) Let $x \in f^{-1}(\bigcup_{i=1}^{n} F_i)$

$\Rightarrow f(x) \in \bigcup_{i=1}^{n} F_i \Rightarrow f(x) \in F_i$ for some $i \in 1 \div n$

$\Rightarrow x \in f^{-1}(F_i)$ for some $i \in 1 \div n$

$\Rightarrow x \in f^{-1}(F_i) \cup$ anything

$\Rightarrow x \in \bigcup_{i=1}^{n} f^{-1}(F_i) \Rightarrow \boxed{f^{-1}(\bigcup_{i=1}^{n} F_i) \subseteq \bigcup_{i=1}^{n} f^{-1}(F_i)}$

2) Let $x \in \bigcup_{i=1}^{n} f^{-1}(F_i)$

$\Rightarrow x \in f^{-1}(F_i)$ for some $i \in 1 \div n$

$\Rightarrow f(x) \in F_i$ for some $i \in 1 \div n$

$\Rightarrow f(x) \in \bigcup_{i=1}^{n} F_i \Rightarrow x \in f^{-1}(\bigcup_{i=1}^{n} F_i)$

$\Rightarrow \boxed{\bigcup_{i=1}^{n} f^{-1}(F_i) \subseteq f^{-1}(\bigcup_{i=1}^{n} F_i)}$

1) & 2) $\Rightarrow f^{-1}(\bigcup_{i=1}^{n} F_i) = \bigcup_{i=1}^{n} f^{-1}(F_i)$

This way one can prove for arbitrary collection:

$$f^{-1}(\bigcup_{\alpha} F_\alpha) = \bigcup_{\alpha} f^{-1}(F_\alpha)$$

Problem 3.1 : Prove that for any function $f : A \mapsto B$, if $E, F \subseteq A$, then $f(E \cap F) \subseteq f(E) \cap f(F)$.

Let $y \in f(E \cap F) \Rightarrow y = f(x)$ for $x \in E \cap F$

$\Rightarrow \begin{cases} x \in E \Rightarrow y = f(x) \in f(E) \\ \underline{\text{and}}\; x \in F \Rightarrow y = f(x) \in f(F) \end{cases}$

$y \in f(E)$ and $y \in f(F)$

$\Rightarrow y \in f(E) \cap f(F)$

$\Rightarrow f(E \cap F) \subseteq f(E) \cap f(F)$

(If $f(E \cap F) = \phi \Rightarrow \phi \subseteq f(E) \cap f(F)$)

ϕ set is a subset of any set.

Problem 3.2 : Prove that if f is injective, we have $f(E \cap F) = f(E) \cap f(F)$.

Let $y \in f(E) \cap f(F)$ (i.e. suppose $f(E) \cap f(F) \neq \phi$)

$\Rightarrow y \in f(E) \Rightarrow \exists\, x_1 \in E$ s.th $y = f(x_1)$

and $y \in f(F) \Rightarrow \exists\, x_2 \in F$ s.th. $y = f(x_2)$

But f-injective $\Rightarrow f(x_1) = f(x_2)$ implies $x_1 = x_2$

$\Rightarrow x = x_1 = x_2 \in E$ and $x = x_1 = x_2 \in F$

$\Rightarrow x \in E \cap F$ and $y = f(x)$

$\Rightarrow y \in f(E \cap F) \Rightarrow f(E) \cap f(F) \subseteq f(E \cap F)$

But by 3.1 $\Rightarrow f(E) \cap f(F) = f(E \cap F)$

Problem 3.3 : Is it true that for arbitrary function $f : A \mapsto B$ we have $f(E \cap F) = f(E) \cap f(F)$? Prove your answer.

No : Let $f(x) : \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$

Let $E = \{-1\}$ $F = \{1\}$

$E \cap F = \phi$ $f(E \cap F) = \phi$

$f(E) = \{1\}$ $f(F) = \{1\}$ $f(E) \cap f(F) = \{1\}$

$\{1\} \neq \phi$

Problem 4 : Let $S = \{a + b\sqrt{5} \ / \ a, b \in \mathbb{Q}\}$.

Problem 4.1 : Prove that if $a_1 + b_1\sqrt{5} = a_2 + b_2\sqrt{5}$, and $a_1, a_2, b_1, b_2 \in \mathbb{Q}$, then $a_1 = a_2$ and $b_1 = b_2$.

Proof: $a_1 + b_1\sqrt{5} = a_2 + b_2\sqrt{5}$

$\Rightarrow a_1 - a_2 = (b_2 - b_1)\sqrt{5}$

Proof by contr.: Suppose $b_1 \neq b_2$

$\Rightarrow b_2 - b_1 \neq 0$

$\Rightarrow \sqrt{5} = \dfrac{a_1 - a_2}{b_2 - b_1}$ & $\dfrac{a_1 - a_2}{b_2 - b_1} \in \mathbb{Q}$

But $\sqrt{5} \notin \mathbb{Q}$ $\Rightarrow$ contradiction

$\Rightarrow b_1 = b_2$ $\Rightarrow a_1 - a_2 = 0 \cdot \sqrt{5} = 0$

$\Rightarrow a_1 = a_2$.

No need to prove, but:

Proof that $\sqrt{5} \notin \mathbb{Q}$: Suppose $\sqrt{5} = \frac{m}{n} \in \mathbb{Q}$, and $(m,n) = 1$ $\Rightarrow 5 = \frac{m^2}{n^2} \Rightarrow 5n^2 = m^2$

$\Rightarrow 5 / m^2$, but 5-prime $\Rightarrow 5/m \Rightarrow m = 5m_1$

$\Rightarrow 5n^2 = 25m_1^2$ $n^2 = 5m_1^2 \Rightarrow 5/n$

$\Rightarrow 5/(m,n)$, but $(m,n) = 1$ $\Rightarrow$ contradiction

Problem 4.2 : Prove that S is a subfield of $\mathbb{R}$ (i.e., prove that if $x_1, x_2 \in S$, then $x_1 + x_2$ and $x_1 x_2 \in S$; also if $x \neq 0, x \in S$, then $\frac{1}{x} \in S$).

Let $x_1, x_2 \in S \Rightarrow x_1 = a_1 + b_1\sqrt{5}$
$x_2 = a_2 + b_2\sqrt{5}$, $a_1, a_2, b_1, b_2 \in \mathbb{Q}$

$\Rightarrow x_1 + x_2 = (a_1 + a_2) + (b_1 + b_2)\sqrt{5}$, and $a_1 + a_2 \in \mathbb{Q}$
$b_1 + b_2 \in \mathbb{Q}$

$\Rightarrow x_1 + x_2 \in S$

$x_1 x_2 = (a_1 a_2 + 5b_1 b_2) + (a_1 b_2 + a_2 b_1)\sqrt{5}$,
and $a_1 a_2 + 5b_1 b_2 \in \mathbb{Q}$, $a_1 b_2 + a_2 b_1 \in \mathbb{Q}$ $\Rightarrow x_1 x_2 \in S$

Let $x \in S$ $x \neq 0$ $\Rightarrow$ $x = a + b\sqrt{5}$, $a, b \in \mathbb{Q}$

$\dfrac{1}{x} = \dfrac{a}{a^2 - 5b^2} - \dfrac{b}{a^2 - 5b^2} \cdot \sqrt{5}$

$a^2 - 5b^2 \neq 0$: Proof: Suppose $a^2 = 5b^2$. If $b = 0 \Rightarrow a = 0$
$\Rightarrow x = 0$ – contr. $\Rightarrow b \neq 0 \Rightarrow \dfrac{a^2}{b^2} = 5 \Rightarrow \left|\dfrac{a}{b}\right| = \sqrt{5}$
but $\left|\frac{a}{b}\right| \in \mathbb{Q}$, $\sqrt{5} \notin \mathbb{Q}$ $\Rightarrow$ contradiction

Since $a^2 - 5b^2 \neq 0$

$$\Rightarrow \frac{a}{a^2-5b^2}, \frac{-b}{a^2-5b^2} \in \mathbb{Q}$$

$$\Rightarrow \frac{a}{a^2-5b^2} - \frac{b}{a^2-5b^2} \cdot \sqrt{5} \in S$$

Also $\left(\frac{a}{a^2-5b^2} - \frac{b}{a^2-5b^2} \cdot \sqrt{5}\right) \cdot \left(a + \sqrt{5}\, b\right) =$

$$= \frac{a^2 - 5b^2}{a^2 - 5b^2} = 1$$

$$\Rightarrow \frac{a}{a^2-5b^2} - \frac{b}{a^2-5b^2} \cdot \sqrt{5} = \frac{1}{x} \in S$$

Problem 4.3 : Establish a bijection between $\mathbb{Q}\times\mathbb{Q}$ and S. Prove your answer.

Let $f: \mathbb{Q} \times \mathbb{Q} \to S$
$(a, b) \to a + b\sqrt{5}$

(1) f - surjection - obviously:
If $s \in S \Rightarrow s = a + b\sqrt{5}$, $a, b \in \mathbb{Q}$
$\Rightarrow s = f(a, b)$

(2) f - injective: If $f_1 = f_2 \Rightarrow s_1 = s_2$
$\Rightarrow a_1 + b_1\sqrt{5} = a_2 + b_2\sqrt{5}$
$\Rightarrow a_1 = a_2$ and $b_1 = b_2$
$\Rightarrow (a_1, b_1) = (a_2, b_2)$
$\Rightarrow f$ - injective $\Rightarrow f$ - bijection

Problem 5.1 : Prove that $0 < \frac{2}{5} < 1$.

Proof: By Th: 2.1.8 $2 > 0$ and $5 > 0$ and $3 > 0$
We have $5 \cdot (\frac{1}{5}) = 1 > 0$ and $5 > 0$
$\Rightarrow$ By Th. 2.1.10 $\Rightarrow \frac{1}{5} > 0$
$2 > 0$ and $\frac{1}{5} > 0 \Rightarrow \frac{2}{5} = 2 \cdot \frac{1}{5} > 0$

Also: $1 - \frac{2}{5} = \frac{3}{5} = 3 \cdot \frac{1}{5} > 0$ since $3 > 0$ and $\frac{1}{5} > 0$

$\Rightarrow \frac{2}{5} < 1 \Rightarrow$ $\frac{2}{5} > 0$, $\frac{2}{5} < 1$ $\Rightarrow 0 < \frac{2}{5} < 1$

Problem 5.2 : Prove that $0 < (\frac{2}{5})^2 < \frac{2}{5}$.

Proof: $\frac{2}{5} > 0$ mult/. $\frac{2}{5} > 0$ By Th. 2.1.7 c
$\Rightarrow \frac{2}{5} \cdot \frac{2}{5} > \frac{2}{5} \cdot 0$
$\Rightarrow (\frac{2}{5})^2 > 0 \Rightarrow 0 < (\frac{2}{5})^2$

Also from 5.1 $1 > \frac{2}{5} \Rightarrow \frac{2}{5} \cdot 1 > \frac{2}{5} \cdot \frac{2}{5}$
$\Rightarrow \frac{2}{5} > (\frac{2}{5})^2 \Rightarrow (\frac{2}{5})^2 < \frac{2}{5}$

$\Rightarrow 0 < (\frac{2}{5})^2 < \frac{2}{5}$

Problem 6.1 : You are rolling a standard 6-sided die (one of a pair of dice) and you never stop (i.e., you are rolling your die over and over forever). You make a list of the numbers that come up. Is the set of possible lists countable? Prove your answer.

Let S is the set of all your possible outcomes.
S consists of infinite sequences with $1 \div 6$ allowed in every term.
$\Rightarrow S$ has a subset T consisting of the infinite sequences with 1 and 2 (only) allowed in every term.
T is bijective to the set B of sequences with L, R allowed in every term.
i.e. $T \rightleftharpoons B \Rightarrow T$ is uncountable, since B is uncountable.
T is a subset of $S \Rightarrow S$ - uncountable (by Th 1.3.9 in book)

Problem 6.2 : Suppose now your friend is also rolling a die, but he always stops rolling at some point in time. Is the set of your friend's possible lists countable? Prove your answer.

Your friend stops at some time.
Let A_n be the subset of outcomes corresponding to the case when your friend stops after n rollings.
We have that M (the set of your friends outcomes)

$$M = \bigcup_{n=1}^{\infty} A_n$$

(Since he stops rolling at some time, n, but this n is not fixed, but can be any $\mathbb{N}$).
Now we have $|A_n| = 6^n$ (sim. to pr. 11)
$\Rightarrow A_n$ - finite $\Rightarrow A_n$ - countable.
$\Rightarrow M$ - countable union of countable sets $\Rightarrow M$ - countable

One can prove $|A_n| = 6^n$ by induction, but we are not taking points off for not proving this.

Proof: ① set $(A_1) = \{1, 2, \vdots, 6\}$ $\Rightarrow$ cardinality $|A_1| = 6$

② suppose set A_k has cardinality $|A_k| = 6^k$

③ The set A_{k+1} consists of finite sequences with $k+1$ terms

$$a_1 a_2 \dots a_k a_{k+1}, \quad a_{k+1} \in 1 \div 6$$

$\Rightarrow$ We have the possibilities

$a_1 a_2 \dots a_k 1$

$a_1 a_2 \dots a_k 2$ $\Rightarrow$ $6 \times$ times the possibilities of the $a_1 . a_2 \dots . a_k$

$\vdots$

$a_1 a_2 \dots a_k 6$

$$\Rightarrow |A_{k+1}| = 6 \times |A_k| = 6 . 6^k = 6^{k+1}$$

①, ②, ③ $\Rightarrow$ $\forall n \; |A_n| = 6^n$