MAT 320 Fall 2007 Review for Final

Note: Final is cumulative, so use the Midterm 2 Review and the Midterm 1 Practice Exam as well as the material below.

- Theorem: be able to apply
- *Theorem*: and know what goes into the proof
- **Theorem**: and be able to prove.

§7.1 Understand the parallelism between the definition of "f is Riemann integrable on $[a,b]$ with integral L" and, for example, "the sequence (a_n) is convergent with limit L." Basic: **Theorem 7.1.2** The integral is unique. Understand examples (c) and (d) on p.198, and understand the elementary **Theorem 7.1.4**. Also *Theorem 7.1.5*, and review Example 7.1.6 (Thomae's function on $[0,1]$ is in $\mathcal{R}([0,1])$.

§7.2 *Theorem 7.2.1* (Cauchy Criterion) important because it gives a definition of "f integrable on $[a,b]$" that does not involve the value of $\int_a^b f$. **Theorem 7.2.3 - "Squeeze Theorem"** used in proof of **Theorem 7.2.6**: If f is continuous on $[a,b]$ then $f \in \mathcal{R}([a,b])$. *Theorem 7.2.7*: If f is monotone on $[a,b]$ then $f \in \mathcal{R}([a,b])$. Theorem 7.2.8 (Additivity Theorem) etc.

§7.3 *Theorem 7.3.1 Fundamental Theorem, I.* Understand where all the hypotheses are used; in particular understand Example 7.3.2(e). **Theorem 7.3.4**: if $f \in \mathcal{R}([a,b])$, then the function $x \mapsto \int_a^x f$ is continuous on $[a,b]$; elementary once you have 7.1.5 and additivity. **Theorem 7.3.5 Fundamental Theorem, II**. *Theorem 7.3.6* is a corollary.

§8.1 Definition 8.1.1: convergence $(f_n) \to f$ is defined *pointwise*. Understand the difference from Definition 8.1.4: uniform continuity $(f_n) \rightrightarrows f$ (book uses double arrow). Understand why the convergence in Examples 8.1.2 (a,b) is not uniform. Understand the "uniform norm" $\|f-g\|_D = \sup_{x\in D}|f(x)-g(x)|$ as a measure of the distance from f to g, and in terms of this norm understand **Theorem 8.1.10**: a Cauchy criterion allowing us to prove (f_n) converges uniformly without *a priori* knowing what the limit is. Obviously useful.

§8.2 This section contains three important theorems describing how continuity, integrability and differentiability behave under *uniform* limits. They are all proved by 3ϵ arguments. (Review the Examples 8.2.1 (a,b,c) to see what can go wrong when convergence is not uniform). **Theorem 8.2.2**: a uniform limit of continuous functions is continuous. *Theorem 8.2.3* and **Theorem 8.2.4**.

§9.1 Understand that an infinite sum interpreted literally does not make sense, and gets meaning as the limit of the sequence of partial sums, where it is amenable to ϵ, N analysis. Go back to section 3.7 and make sure you know how to show that $\sum_0^\infty ar^n = a/(1-r)$ when $|r| < 1$, and diverges otherwise. Know the "nth term test," the comparison test and the Cauchy criterion for series. You should know an elementary proof that $\sum_1^\infty \frac{1}{n}$ diverges and that $\sum_1^\infty (-1)^{n+1}\frac{1}{n}$ converges.

Know the definition of "$\sum x_n$ is absolutely convergent," and **Theorem 9.1.2**: an absolutely convergent series is convergent. Understand the definition of "rearrangement" (9.1.4) and the *Rearrangement Theorem 9.1.5*.

§9.2 Understand the *Root Test*, the **Ratio Test** and the **Integral test** - remember that f must be positive and decreasing. Know the applications to the "p-series" $\sum_{n=1}^\infty (1/n^p)$.

§9.3 Understand the **Alternating Series Test**.

§9.4 An infinite sum of functions $\sum_{n=1}^\infty f_n$ means the limit (if it exists) of the sequence of partial-sum functions $s_n = f_0 + \cdots + f_n$. Similarly, the sum $\sum_{n=1}^\infty f_n$ converges uniformly to f if $(s_n) \rightrightarrows f$. The theorems of §8.2 translate into theorems about series: *Theorems 9.4.2, 9.4.3, 9.4.4*; as does the Cauchy Criterion (9.4.5); its corollary is the *Weierstrass M-test 9.4.6*.

There is a special and important analysis for power series. Know the extreme examples $\sum_{n=0}^\infty n!x^n$ and $\sum_{n=0}^\infty (x^n/n!)$ and remember that $\sum_{n=0}^\infty x^n$ is a geometric series converging for $|x| < 1$ and diverging otherwise. Understand the definition of "limit superior" of a bounded sequence (b_n), because the radius of convergence R of the series $\sum_{n=0}^\infty a_n x^n$ is defined in terms of $\limsup(|a_n|^{1/n})$ -essentially, its reciprocal: Definition 9.4.8 and **Theorem 9.4.9**. This theorem is overkill for series for which $R = \lim |a_n/a_{n+1}|$ exists: then that R is the radius of convergence (Exercise 5).

Theorem 9.4.10: a power series $\sum a_n x^n$ with radius of convergence R converges uniformly on any closed, bounded interval $K \subset (-R, R)$. **Theorems 9.4.11** and **9.4.12** then follow from the theorems of section 8.2.