MAT 320 Fall 2007  Review for Final

Note: Final is cumulative, so use the Midterm 2 Review and the Midterm 1 Practice Exam as well as the material below.

- Theorem: be able to apply
- Theorem: and know what goes into the proof
- Theorem: and be able to prove.

§7.1 Understand the parallelism between the definition of “\( f \) is Riemann integrable on \([a,b]\) with integral \( L \)” and, for example, “the sequence \((a_n)\) is convergent with limit \( L \)” Basic: Theorem 7.1.2 The integral is unique. Understand examples (c) and (d) on p.198, and understand the elementary Theorem 7.1.4. Also Theorem 7.1.5, and review Example 7.1.6 (Thomae’s function on \([0,1]\) is in \( R([0,1]) \)).

§7.2 Theorem 7.2.1 (Cauchy Criterion) important because it gives a definition of “\( f \) integrable on \([a,b]\)” that does not involve the value of \( \int_a^b f \). Theorem 7.2.3 - “Squeeze Theorem” used in proof of Theorem 7.2.6: If \( f \) is continuous on \([a,b]\) then \( f \in R([a,b]) \). Theorem 7.2.7: If \( f \) is monotone on \([a,b]\) then \( f \in R([a,b]) \). Theorem 7.2.8 (Additivity Theorem) etc.

§7.3 Theorem 7.3.1 Fundamental Theorem, I. Understand where all the hypotheses are used; in particular understand Example 7.3.2(e). Theorem 7.3.4: if \( f \in R([a,b]) \), then the function \( x \mapsto \int_a^x f \) is continuous on \([a,b]\); elementary once you have 7.1.5 and additivity. Theorem 7.3.5 Fundamental Theorem, II. Theorem 7.3.6 is a corollary.

§8.1 Definition 8.1.1: convergence \((f_n) \to f\) is defined pointwise. Understand the difference from Definition 8.1.4: uniform continuity \((f_n) \Rightarrow f\) (book uses double arrow). Understand why the convergence in Examples 8.1.2 (a,b) is not uniform. Understand the “uniform norm” \( \|f-g\|_D = \sup_{x \in D} |f(x) - g(x)| \) as a measure of the distance from \( f \) to \( g \), and in terms of this norm understand Theorem 8.1.10: a Cauchy criterion allowing us to prove \((f_n)\) converges uniformly without a priori knowing what the limit is. Obviously useful.

§8.2 This section contains three important theorems describing how continuity, integrability and differentiability behave under uniform limits. They are
all proved by $3\epsilon$ arguments. (Review the Examples 8.2.1 (a,b,c) to see what can go wrong when convergence is not uniform). **Theorem 8.2.2**: A uniform limit of continuous functions is continuous. **Theorem 8.2.3** and **Theorem 8.2.4**.

§9.1 Understand that an infinite sum interpreted literally does not make sense, and gets meaning as the limit of the sequence of partial sums, where it is amenable to $\epsilon, N$ analysis. Go back to section 3.7 and make sure you know how to show that $\sum_0^\infty ar^n = a/(1 - r)$ when $|r| < 1$, and diverges otherwise. Know the “$n$th term test,” the comparison test and the Cauchy criterion for series. You should know an elementary proof that $\sum_1^\infty \frac{1}{n}$ diverges and that $\sum_1^\infty (-1)^{n+1}\frac{1}{n}$ converges.

Know the definition of “$\sum x_n$ is absolutely convergent,” and **Theorem 9.1.2**: an absolutely convergent series is convergent. Understand the definition of “rearrangement” (9.1.4) and the **Rearrangement Theorem 9.1.5**.

§9.2 Understand the Root Test, the Ratio Test and the Integral test - remember that $f$ must be positive and decreasing. Know the applications to the “$p$-series” $\sum_{n=1}^\infty (1/n^p)$.

§9.3 Understand the Alternating Series Test.

§9.4 An infinite sum of functions $\sum_{n=1}^\infty f_n$ means the limit (if it exists) of the sequence of partial-sum functions $s_n = f_0 + \cdots + f_n$. Similarly, the sum $\sum_{n=1}^\infty f_n$ converges uniformly to $f$ if $(s_n) \Rightarrow f$. The theorems of §8.2 translate into theorems about series: **Theorems 9.4.2, 9.4.3, 9.4.4**; as does the Cauchy Criterion (9.4.5); its corollary is the **Weierstrass M-test 9.4.6**.

There is a special and important analysis for power series. Know the extreme examples $\sum_{n=0}^\infty n!x^n$ and $\sum_{n=0}^\infty (x^n/n!)$ and remember that $\sum_{n=0}^\infty x^n$ is a geometric series converging for $|x| < 1$ and diverging otherwise. Understand the definition of “limit superior” of a bounded sequence $(b_n)$, because the radius of convergence $R$ of the series $\sum_{n=0}^\infty a_n x^n$ is defined in terms of $\limsup(|a_n|^{1/n})$ - essentially, its reciprocal: Definition 9.4.8 and **Theorem 9.4.9**. This theorem is overkill for series for which $R = \lim |a_n/a_{n+1}|$ exists: then that $R$ is the radius of convergence (Exercise 5).

**Theorem 9.4.10**: A power series $\sum a_n x^n$ with radius of convergence $R$ converges uniformly on any closed, bounded interval $K \subset (-R, R)$. **Theorems 9.4.11** and **9.4.12** then follow from the theorems of section 8.2.