1. (25 points) The functions $f$ and $g$ are continuous on the interval $[a, b]$ and differentiable on $(a, b)$. If $f(a) = g(a)$ and $f(b) = g(b)$, prove that there is a point $c$, with $a < c < b$, where $f'(c) = g'(c)$.

- Apply Rolle’s Theorem to $f - g$.

2. (25 points) The function $\sin x$ is infinitely differentiable on $\mathbb{R}$, and its derivatives cycle through $\cos x, -\sin x, -\cos x, \sin x$. Let $P_k(x)$ be the $k$-th Taylor polynomial, based at 0, for $\sin x$. Prove that

$$\lim_{k \to \infty} P_k(x) = \sin x$$

for every $x \in \mathbb{R}$.

- Choose $x$. By Taylor’s Theorem, there exists $c$ between 0 and $x$ such that

$$\sin x - P_k(x) = \frac{f^{(k+1)}(c)x^{k+1}}{(k+1)!}.$$ 

Since $|f^{(k+1)}(c)| \leq 1$, we have $|\sin x - P_k(x)| \leq |x|^{k+1}/(k+1)!$. The proof then follows from $\lim_{n \to \infty} a^n/n! = 0$, true for any $a \in \mathbb{R}$.

3. (25 points) The function $f$ is continuous and twice differentiable on the interval $[a, b]$, with both derivatives positive there: $f'(x) > 0$ and $f''(x) > 0$ for every $a \leq x \leq b$. Suppose that $f(a) < 0$ and $f(b) > 0$. Prove that the tangent line to the graph of $f$ at $b$ intersects the $x$-axis at a point in the interval $[a, b]$. (This corresponds to starting Newton’s Method with $x_0 = b$, and then $x_1$ is that intersection point, but no knowledge of Newton’s Method is necessary for this question).

- Since Let $x$ be the intersection point; since $f(b) > 0$ and $f'(b) > 0$, we know $x < b$. To show $a < x$, argue as follows: by the Intermediate Value Theorem, there is $c, a < c < b$, with $f(c) = 0$. By the Mean Value Theorem, there is $k, c < k < b$, with $f'(k) = (f(b) - f(c))/(b - c)$, or $f'(k) = f(b)/(b - c)$. The equation of the
tangent line at \((b, f(b))\) is \(y = f(b) + f'(b)(x - b)\), and it intersects the \(x\)-axis when \(x - b = -f(b)/f'(b)\). By hypothesis \(f''(x)\) is always positive, so \(f'(k) < f'(b)\). Finally

\[
b - x = f(b)/f'(b) < f(b)/f'(k) = b - c.
\]

It follows that \(c < x\), and therefore \(a < x\).

4. (25 points) The Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, \(\ldots\) are defined by the properties \(F_0 = F_1 = 1\) and \(F_{n+1} = F_n + F_{n-1}\) for \(n \geq 2\). Define a sequence \((x_n)\) by

\[
x_n = \frac{F_{n+1}}{F_n}.
\]

Prove that this sequence converges, and calculate its limit.

- Notice first that \(x_{n+1} = 1 + 1/x_n\). We first use this to show \(x_n \geq 3/2\) as soon as \(n \geq 1\): the \(x_n\) are all < 2 (since \(F_{n+1} = F_n + F_{n-1} < 2F_n\)); so \(x_{n+1} = 1 + 1/x_n > 1 + 1/2\). Next we use it to show the sequence \((x_n)\) is contractive:

\[
|x_{n+1} - x_n| = |1 + 1/x_{n-1} - 1/x_n| = \frac{|x_{n-1} - x_n|}{x_n x_{n-1}} \leq (4/9)|x_{n-1} - x_n|
\]

using the first calculation. It follows that the sequence converges to a limit \(L\). Substituting \(L\) in \(x_{n+1} = 1 + 1/x_n\) and remembering that the \(x_n\) are all positive leads to \(L = (1/2)(1 + \sqrt{5})\).