Correction of HW4

Exercise 1. Page 67, #6a.

Proof.

By the sum rule, the limit of $(2+1/n)$ is equal to 2. By the product rule, the limit of $(2+1/n)^2$ is $2.2=4$.

□

Exercise 2. Page 67, #9.

Proof. One has $y_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ (multiply the numerator and denominator by the conjugate quantity).

But now one has $0 \leqslant y_n \leqslant \frac{1}{\sqrt{n}}$. Therefore if one proves that $(1/\sqrt{n})$ converges to zero, the squeeze theorem implies that (y_n) converges itself to zero.

Fix any $\varepsilon > 0$, then by the archimedean property there exists a natural number K such that $K > \varepsilon^2$, but this implies that for any $n \geqslant K$ one has $n > \varepsilon^2$, implying $\frac{1}{\sqrt{n}} < \varepsilon$, thus we proved that $(1/\sqrt{n})$ converges to zero, and hence (y_n) converges to zero.

Now $\sqrt{n}\, y_n = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n}}.\frac{1}{\sqrt{(1+1/n)}+1} = \frac{1}{1+\sqrt{1+\frac{1}{n}}}$. By the square root theorem $\sqrt{1+\frac{1}{n}}$ converges to 1. By the quotient theorem (which applies because the limit of the denominator is nonzero), one knows that $\sqrt{n}.y_n$ converges to $1/2$.

□

Exercise 3. Page 67, #21.

Proof. Pick any $\varepsilon > 0$.

Since (x_n) is convergent to a limit x, we know the existence of a natural number K such that for any $n \geqslant K$ one has $|x_n - x| < \varepsilon/2$. We also know the existence of another natural number M' such that for any $n \geqslant M'$ one has $|x_n - y_n| < \varepsilon/2$.

Now for any $n \geqslant L = \max\{K, M'\}$ one has by the triangle inequality:

$$|y_n - x| \leqslant |y_n - x_n| + |x_n - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Thus we proved that (y_n) converges, to the same limit x.

□

Exercise 4. Page 74, #1.

Proof. Let's prove that for any $n \geqslant 1$ one has $4 \leqslant x_n \leqslant 8$.(Make a drawing to guess these bounds!)

This is true for $n=1$ (because $x_1 = 8$). Assume it is true for k: then one has $x_{k+1} = \frac{1}{2}x_k + 2 \leqslant \frac{1}{2}8 + 2 = 6 \leqslant 8$, and also $\frac{1}{2}x_k + 2 \geqslant \frac{4}{2} + 2 = 4$ so this is true for x_{k+1}.

Now let us prove that the function $f(x) = \frac{1}{2}x + 2$ is such that $f(x) < x$ on the interval $(4, 8]$: indeed $f(x) < x$ is equivalent to $\frac{1}{2}x + 2 < x$, which is equivalent to $2 < \frac{1}{2}x$ or simply $x > 4$.

Therefore since any x_n belongs to that interval, one has that $x_{n+1} = f(x_n) < x_n$, and thus our sequence is strictly decreasing. Since it is also bounded below, we know that it must converge to a limit x.

Now the limit x must satisfy $x = \frac{1}{2}x + 2$, which is equivalent to $x = 4$, so the limit is 4. $\square$

Exercise 5. Page 74, #4.

Proof. Let's prove that for any $n \geqslant 1$ one has $0 \leqslant x_n \leqslant 2$. (Again make a drawing to guess this). This is true for $n = 1$ because $x_1 = 1$. Assume it is true for k:

then one has $2 \leqslant 2 + x_k \leqslant 4$ and thus $0 \leqslant \sqrt{2} \leqslant \sqrt{2 + x_k} \leqslant \sqrt{4} = 2$, so it is true for $k + 1$.

Now let us prove that on the interval $[0, 2]$ the function $f(x) = \sqrt{2 + x}$ satisfies $f(x) \geqslant x$.But since $f(x) > 0$ on this interval, so is $\sqrt{2 + x} + x$, thus one has $f(x) - x = \frac{2}{\sqrt{2 + x} + x} \geqslant 0$ (multiply numerator and denominator by the conjugate quantity, which is nonzero).

Since any x_n belongs to that interval one has $x_{n+1} = f(x_n) \geqslant x_n$, so the sequence is increasing. It is also bounded by 2, so it is convergent to a real number x by our theorem on convergence of monotone bounded sequences. Now the limit x must satisfy $x = \sqrt{2 + x}$, which is equivalent to $\left(x \geqslant 0 \text{ and } x^2 = 2 + x\right)$, which is equivalent to $(x = 2)$ (notice that $x^2 - x - 2 = (x - 2).(x + 1)$ and we want the positive root).

Thus the sequence converges to 2.

$\square$