MAT 319 Spring 2015
Notes on series and Ratio Test

Proposition 1: A convergent sequence is a Cauchy sequence.
Proof: This is Ross, Lemma 10.9

Definition: The series $\sum_1^\infty a_k$ converges means that the sequence $(s_n = \sum_1^n a_k)$ of *partial sums* is a convergent sequence. (Ross, section 14.2).

Proposition 2: (Cauchy criterion) the series $\sum_1^\infty a_k$ converges if and only if for every $\epsilon > 0$ there exists an N such that $m, n > N, m > n$ implies $\sum_{n+1}^m a_k < \epsilon$.

Proof: This is Ross, Theorem 14.4.

Proposition 3: If the series $\sum_1^\infty a_k$ converges then $\lim a_k = 0$.

Proof: Need to show for any $\epsilon > 0$ there exists an index N such that if $n > N$ then $|a_n| < \epsilon$. If the series converges, it satisfies the Cauchy criterion: there exists an N' such that if $m, n > N'$ (and $m \geq n$) then $|\sum_{n+1}^m a_k| < \epsilon$. Take $N = N' + 1$. If $n > N$ then $n - 1 > N'$ and $|\sum_{n-1+1}^m a_k| < \epsilon$, i.e. $|\sum_n^m a_k| < \epsilon$. In particular, take $m = n$. Then $|a_n| = |\sum_n^m a_k| < \epsilon$, as required. [This is better than the argument I gave in class, which required proving first that $\lim a_k$ exists.]

Proposition 4 (Comparison Test): Suppose $\sum a_k$ is a convergent series with positive terms (every $a_k \geq 0$). Then if the terms of a series $\sum b_k$ satisfy $|b_k| \leq a_k$ for every k, the series $\sum b_k$ converges.

Proof. We use the Cauchy criterion. For every $\epsilon > 0$ there exists an index N such that $m, n > N$ implies $\sum_{n+1}^m a_k < \epsilon$. Suppose then $m, n > N$; then $|\sum_{n+1}^m b_k| \leq \sum_{n+1}^m |b_k| \leq \sum_{n+1}^m a_k < \epsilon \quad (*)$, so $\sum b_k$ satisfies the Cauchy criterion and therefore converges. Triangle inequality used in $(*)$.

Definition: A series $\sum a_k$ *converges absolutely* means that $\sum |a_k|$ converges.

Proposition 5: If a series converges absolutely, it converges.

Proof: Since $a_k \leq |a_k|$ this follows from the Comparison Test.

Proposition 6 (Ratio Test): Suppose a series $\sum a_k$ of non-zero terms satisfies $\lim |\frac{a_{n+1}}{a_n}| = R$. Then if $R < 1$ the series $\sum a_k$ converges absolutely, and if $R > 1$ it diverges.

Proof. First suppose $R < 1$. Let $\epsilon = \frac{1}{2}(1 - R)$. Note that $\frac{1}{2}(1 - R) > 0$ so there exists an index N such that if $n > N$ then

$$||\frac{a_{n+1}}{a_n}| - R| < \frac{1}{2}(1 - R),$$

which means

$$R - \frac{1}{2}(1 - R) < |\frac{a_{n+1}}{a_n}| < R + \frac{1}{2}(1 - R).$$

Set $\rho = R + \frac{1}{2}(1 - R) = \frac{1}{2}(1 + R)$ and note that $\rho < 1$. In particular,

$$|\frac{a_{N+2}}{a_{N+1}}| < \rho \text{ so } |a_{N+2}| < \rho |a_{N+1}|$$

$$|\frac{a_{N+3}}{a_{N+2}}| < \rho \text{ so } |a_{N+3}| < \rho |a_{N+2}| < \rho^2 |a_{N+1}|$$

$$\cdots$$

$$|\frac{a_{N+i}}{a_{N+i-1}}| < \rho \text{ so } |a_{N+i}| < \rho |a_{N+i-1}| < \ldots < \rho^{i-1} |a_{N+1}|$$

$$\cdots.$$

With the N we have obtained, let us write

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{N} |a_k| + \sum_{k=N+1}^{\infty} |a_k|.$$

The first sum is finite. We can apply the Comparison Test to the second sum, which equals $|a_{N+1}| + |a_{N+2}| + \cdots$. The series $\sum_{k=N+1}^{\infty} |a_k|$ is equal to $\sum_{i=1}^{\infty} |a_{N+i}|$ (just rewriting the indices). Since $|a_{N+i}| \leq \rho^{i-1}|a_{N+1}|$ (note that this holds for $i = 1$ also), and $\sum_{i=1}^{\infty} \rho^{i-1}|a_{N+1}|$ is a geometric series converging to

$$|a_{N+1}| \sum_{i=1}^{\infty} \rho^{i-1} = |a_{N+1}| \sum_{i=0}^{\infty} \rho^i = \frac{|a_{N+1}|}{1 - \rho}$$

the Comparison Test tells us that $\sum_{k=N+1}^{\infty} |a_k|$ converges. Throwing in the finite sum $\sum_{k=1}^{N} |a_k|$ exhibits $\sum_{k=1}^{\infty} |a_k|$ as a convergent sequence, as was to be shown.

Now suppose $R > 1$, and take $\epsilon = \frac{1}{2}(R - 1)$. Arguing as before, we can find an N such that if $n > N$ then

$$R - \frac{1}{2}(R - 1) < |\frac{a_{n+1}}{a_n}| < R + \frac{1}{2}(R - 1).$$

Set $\rho = R - \frac{1}{2}(R - 1) = \frac{1}{2}(R + 1)$ and note that $\rho > 1$. In particular,

$$|\frac{a_{N+2}}{a_{N+1}}| > \rho \text{ so } |a_{N+2}| > \rho |a_{N+1}|$$

$$\cdots$$

$$|\frac{a_{N+i}}{a_{N+i-1}}| > \rho \text{ so } |a_{N+i}| > \rho |a_{N+i-1}| > \ldots > \rho^{i-1} |a_{N+1}|$$

$$\cdots.$$

If $\sum a_k$ converges, then (Proposition 3) $\lim a_k = 0$. But here $\lim_{i\to\infty} |a_{N+i}| > \lim_{i\to\infty} \rho^{i-1}|a_{N+1}| = \infty$ since $\rho > 1$. Since the terms indexed beyond $N + 1$ are going to ∞ in absolute value, they have no chance of going to zero, so the sum does not converge.