MAT 319 Spring 2015
Notes on convergence of sequences

Proposition 1: Every monotone, bounded sequence converges.
Proof: This is Ross, Theorem 10.2.

Proposition 2: Every sequence has a monotone subsequence.
Proof: This is Ross, Theorem 11.3.

Definition: (s_n) is a Cauchy sequence if for every $\epsilon > 0$ there exists a natural number N such that if $m, n > N$ then $|s_m - s_n| < \epsilon$. (Ross, Definition 10.8)

Proposition 3: Every Cauchy sequence converges.
Proof: We first establish (A): A Cauchy sequence (s_n) is bounded. This means that if (s_n) is Cauchy, then there exists a number M such that $|s_n| \leq M$ for every index n.

- Taking $\epsilon = 1$, the definition of Cauchy sequence tells us that there exists N such that if $n, m > N$ then $|s_n - s_m| < 1$. We use this index N in the rest of the argument. In particular, if $n > N$ then $|s_n - s_{N+1}| < 1$, so $|s_n| = |s_n - s_{N+1} + s_{N+1}| \leq |s_n - s_{N+1}| + |s_{N+1}| < 1 + |s_{N+1}|$. On the other hand, the finite set of terms $s_1, s_2, \ldots, s_N$ have a finite maximum absolute value $R = \max\{|s_1|, |s_2|, \ldots, |s_N|\}$. We can now take $M = \max\{R, 1 + |s_{N+1}|\}$ since if $n \leq N$ then $|s_n| \leq R$, whereas if $n > N$ then $|s_n| < 1 + |s_{N+1}|$.

Next we establish (B): A Cauchy sequence (s_n) has a convergent subsequence.

- By Proposition 2, (s_n) has a monotone subsequence, say $(s_{n_k}) = s_{n_1}, s_{n_2}, s_{n_3}, \ldots$. Since (s_n) is bounded, by (A), any subsequence of s_n is bounded. So (s_{n_k}), being monotone and bounded, converges by Proposition 1 to a limit we will call L.

Now we can prove (C): with the notation above, $\lim s_n = L$. I.e. the whole sequence converges to the limit we have established for the subsequence.

- We need to show that for any $\epsilon > 0$ there exists an index P such that if $n > P$ then $|s_n - L| < \epsilon$. Since (s_n) is Cauchy we know there exists an index P such that if $n, m > P$ then $|s_n - s_m| < \epsilon/2$. Since (s_{n_k}) is a subsequence, there is a $J_1 \geq J$ such that if $j > J_i$ then $n_j > P$. We know from (B) that there exists a (sub)index J_2 such that if $j > J_2$ then $|s_{n_j} - L| < \epsilon/2$. Take any $j > \max\{J_1, J_2\}$, and suppose $n > P$. Then $|s_n - L| = |s_n - s_{n_j} + s_{n_j} - L| \leq |s_n - s_{n_j}| + |s_{n_j} - L|$. Since $j > J_1$ we know that n_j is also $> P$, so the first term is $< \epsilon/2$. Since $j > J_2$ the second term is also $< \epsilon/2$, so their sum is less than ϵ, as was to be shown.