Section 3.3

4)

1. By Burnside's Theorem, there are $\frac{1}{6}(2+2+2) = 1$ equivalence class. This agrees.
2. By Burnside's Theorem again, there are $\frac{1}{6}(3+3+2+2+2+6) = 3$ equivalence classes. This agrees.

5) Label the three vertices of the triangle A, B, and C. Let X be the set of colorings of $\{A, B, C\}$ by three colors. $|X| = 3^3 = 27$.

The identity permutation fixes any coloring, so $|X_e| = 27$.

In order of a coloring to be fixed by (ABC), then A, B, and C must all have the same color. Therefore, the coloring is completely determined by the color of A, and there are three choices for this. $|X_{(ABC)}| = 3$. Similarly, $|X_{(ACB)}| = 3$.

The permutation (BC) fixes exactly those colorings for which B and C have the same color. The color of A is arbitrary. Therefore, the coloring is determined by the colors of A and B; so there are $3^2 = 9$ fixed colorings. $|X_{(BC)}| = 9$. Similarly, $|X_{(AC)}| = |X_{(AB)}| = 9$.

Thus, by Burnside's Theorem, the number of inequivalent colorings is: $\frac{1}{6}(27+3+3+9+9+9) = 10$.

6) Label the vertices of the rectangle, going clockwise A, B, C, D. The symmetries of the rectangle are the identity, the two reflections $(AB)(CD)$ and $(AD)(BC)$, and the rotation $(AC)(BD)$. There are 81 colorings of the vertices by 3 colors. $|X_e| = 81$. $|X_{(AB)(CD)}| = |X_{(AD)(BC)}| = |X_{(AC)(BD)}| = 3^2 = 9$. So by Burside's Theorem, $k = \frac{1}{4}(81+9+9+9) = 27$.

7) We think of the necklace as a regular pentagon where we want to color the vertices with two colors: ruby and diamond. Label the vertices of the pentagon A, B, C, D, and E. The group of symmetries of this figure has ten elements: 5 rotations (including the identity), and 5 reflections. There are 32 different colorings of the pentagon, so $|X_e| = 32$. The four other rotations are $(ABCDE), (ACEBD), (ADBEC), (AEDCB)$, so each one fixes only 2 colorings, the two constant colorings. The five reflections are of the form $(A)(BE)(CD)$; fixing one vertex and pairing up the remaining 4. Thus, they fix $2^3 = 8$ elements since they are each a product of three cycles. Therefore,

$$k = \frac{1}{10}(32 + 4\cdot 2 + 5\cdot 8) = 8$$

9)

a) Two colorings are equivalent if there is a symmetry of the figure, inducing a permutation on the set of vertices, that preserves color. Thus, blue vertices must always be mapped to other blue vertices, and can only be mapped from blue vertices. Thus, the number of blue vertices must remain constant.

b) Six inequivalent colorings are as follows: all blue vertices, all red vertices, 3 blue and 1 red, 3 red and 1 blue, 2 blue and 2 red with the 2 blues at adjacent vertices, 2 blue and 2 red with the 2 blues at opposite vertices.

11) In addition to the 8 group elements listed in example 3.36, there are 8 more gotten by composing with the operation that inverts the output. Using the notation on page 139, this operation sends f_n to f_{15-n} for all $0 \leq n \leq 15$. The first eight still fix the same number of elements as stated on page 150.

The inverting output operation, which I will denote as $f \mapsto \ f$ fixes no function.

The operation that sends $f(x, y) \mapsto \ f(x', y)$ fixes 2^2 functions, as do the operations $f(x, y) \mapsto \ f(x, y')$ and $f(x, y) \mapsto \ f(x', y')$.

The operation $f(x, y) \mapsto \ f(y, x)$ again fixes no function, for then we would have to have $f(0, 0) = \ f(0, 0)$. Similarly, by looking at $(1, 0)$, we see that $f(x, y) \mapsto \ f(y', x')$ fixes no function. The operations $f(x, y) \mapsto \ f(y, x')$ and $f(x, y) \mapsto \ f(y', x)$ both fix 2 functions. Therefore, by Burnside's Theorem

$$k = \frac{1}{16}(16 + 4 + 4 + 4 + 8 + 2 + 2 + 8 + 0 + 4 + 4 + 4 + 0 + 2 + 2 + 0) = 4$$

15) The number of colorings that satisfy the requirement of exactly two sides of each color are $\frac{6!}{2!2!2!} = 90$. The identity preserves all of them, $|X_e| = 90$. We will follow the labels of Example 3.34. The elements like (1245) and (1542), which correspond to rotations of 90° and 270° about the center of a face fix no colorings in the set. Indeed, if (1245) fixed a coloring, then faces 1, 2, 4, and 5 would all be the same color. The element (14)(25), which corresponds to a rotation of 180° about the center of a face fixes colorings where faces 1 and 4 are the same, 2 and 5 are the same, and hence 3 and 6 must also be the same. There are $3! = 6$ such colorings, and there are 3 such symmetries.

The symmetries like (14)(26)(35) that correspond to rotation about the centers of edges also fix 6 colorings. There are 6 such symmetries.

The symmetries (135)(246) that correspond to rotation about a vertex fix colorings where faces 1, 3, and 5 are the same color. There are no such colorings in the set we are considering.

Therefore, by Burnside's Theorem, the number of equivalence classes of colorings is

$$k = \frac{1}{24}(90 + 3(6) + 6(6)) = 6$$