MAT 312/AMS 351
Notes and exercises on normal subgroups and quotient groups.

If $H$ is a subgroup of $G$, the equivalence relation $\sim_H$ is defined between elements of $G$ as follows:

$$g_1 \sim_H g_2 \iff \exists h \in H, g_1 = g_2 h.$$  

*Proposition 1.* This is indeed an equivalence relation.

Proof: The three properties: reflexive, symmetric, transitive correspond to the three properties of a subgroup: $H$ contains the identity element $e$ of $G$, $H$ contains inverses of all its elements, $H$ is closed under composition.

- For any $g \in G$, since $e \in H$ and $ge = g$, it follows that $g \sim_H ge = g$, so the relation $\sim_H$ is reflexive.
- If $g_1 \sim_H g_2$, $\exists h \in H, g_1 = g_2 h$. Since $h^{-1}$ must then also belong to $H$, and $g_1 h^{-1} = g_2 h h^{-1} = g_2$, it follows that $g_2 \sim_H g_1$, so the relation $\sim_H$ is symmetric.
- If $g_1 \sim_H g_2$ and $g_2 \sim_H g_3$, then $\exists h \in H, g_1 = g_2 h$, and also $\exists h' \in H, g_2 = g_3 h'$. Since then $hh' \in H$, and $g_1 = g_2 h = (g_3 h') h = g_3 (h' h)$, it follows that $g_1 \sim_H g_3$; so the relation $\sim_H$ is transitive.

For the $\sim_H$ equivalence class of the element $g \in G$ we have the suggestive notation $gH$ (since every element of that equivalence class is $gh$ for some $h \in H$); This equivalence class is called the left $H$-coset of $g$; “left” because $gH$ is obtained by multiplying every element of $H$ on the left by $g$. Note that the left $H$-coset of the identity $e$ is $H$ itself.

*Right $H$-cosets.* In a completely analogous way one can define $g_1 \equiv_H g_2 \iff \exists h \in H, g_1 = hg_2$. A completely analogous argument proves that, since $H$ is a subgroup, the relation $\equiv_H$ is also an equivalence relation. In this case the equivalence class of $g \in G$ is written $Hg$ and called the right $H$-coset of $g$.

Example. Consider $G = S(3)$, the group of permutations of 3 elements, so in cycle notation $G = \{e, (12), (13), (23), (123), (132)\}$; and consider the subgroup $H = \{e, (12)\}$. Since $|G| = 6$ and $|H| = 2$, we expect 3 left $H$-cosets. They are

- $H = \{e, (12)\}$
- $(13)H = \{(13), (13)(12) = (123)\}$
• \((23)H = \{(23), (23)(12) = (132)\}\).

Note that \((123)H = (12)H\) and \((132)H = (23)H\). A coset can have several names!

On the other hand the three right \(H\)-cosets are

- \(H = \{e, (12)\}\)
- \(H(13) = \{(13), (12)(13) = (132)\}\)
- \(H(23) = \{(23), (12)(23) = (123)\}\).

So in general left \(H\)-cosets and right \(H\)-cosets give two different partitions of \(G\). But sometimes the partitions coincide. In this case \(H\) is called a normal subgroup of \(G\). More formally:

**Definition:** The subgroup \(H\) of group \(G\) is called normal if \(gH = Hg\) for every \(g \in G\).

Example 1. With \(G = S(3)\) as above, consider the subgroup \(H = \{e, (123), (132)\}\). Since \(|G| = 6\) and \(|H| = 3\), we expect two left \(H\)-cosets and two right \(H\)-cosets. In either case one coset must be \(H\) itself; so the other one must contain the three remaining elements, namely \(\{(12), (13), (23)\}\). So \(H\) is normal in \(G\). The same thing will happen whenever \(|H| = \frac{1}{2}|G|\); in this case we say that \(H\) is a subgroup of index 2, and we can state the proposition: Every subgroup of index 2 is normal.

Example 2. Consider \(G = A(4)\), the group of all even permutations of 4 elements. In cycle notation,

\[
G = \{e, (123), (124), (132), (134), (142), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}.
\]

(Since exactly half the permutations in \(S(4)\) are even, this \(G\) is an index-2 subgroup of \(S(4)\), and hence a normal subgroup of \(S(4)\)). In \(G\), consider the subgroup \(H = \{e, (12)(34), (13)(24), (14)(23)\}\). Here with \(|G| = 12, |H| = 4\) we expect three cosets.

The left cosets are

- \(H = \{e, (12)(34), (13)(24), (14)(23)\}\)

The right cosets are

- \(H = \{e, (12)(34), (13)(24), (14)(23)\}\)
\( H(123) = \{ (123), (12)(34)(123) = (243), \\
(13)(24)(123) = (142), (14)(23)(123) = (134) \} \)

\( H(124) = \{ (124), (12)(34)(124) = (234), \\
(13)(24)(124) = (143), (14)(23)(124) = (132) \} \)

Note that \( (123)H = H(123) \) and \( (124)H = H(124) \). So \( H \) is a normal subgroup of \( A(4) \).

**Quotient groups.** When \( H \) is a normal subgroup of \( G \), the law of composition of \( G \) induces a composition between \( H \)-cosets which makes this set also into a group. This is called the **quotient group** of \( G \) by \( H \), and written \( G/H \).

**Definition:** For two cosets \( gH \) and \( g'H \) (\( H \) is a normal subgroup of \( G \), but we write them as left cosets for explicitness) we define \( gH \cdot g'H \) to be \( (gg')H \).

**Proposition 2.** This operation is well-defined, and makes the set of cosets into a group.

Proof: First, we need to show the operation is **well-defined** because the result might be different if we had chosen different names (i.e. different representative elements) for the cosets \( gH \) and \( g'H \). So suppose in fact that \( \gamma \in gH \) and \( \gamma' \in g'H \). We need to show that \( \gamma \gamma'H = gg'H \).

What we know is that there is an element \( h \in H \) such that \( \gamma = gh \), and an element \( h' \) such that \( \gamma' = g'h' \). So \( \gamma \gamma'H = (gh)(g'h')H \). Now \( h'H = H \) since \( h' \in H \), and \( g'h'H = g'H = Hg' \) since \( H \) is normal. Furthermore \( hH = H \) since \( h \in H \), so \( hgh'H = Hg'g' = Hg' \), and finally \( \gamma \gamma'H = gHG'H = gg'H \) using normality again.

Next we need to show that this operation satisfies the three conditions required of a group law.

- **Associativity.** \( g_1H \cdot (g_2H \cdot g_3H) = g_1H \cdot (g_2g_3)H = g_1(g_2g_3)H = (g_1g_2)g_3H = (g_1g_2)H \cdot g_3H = (g_1H \cdot g_2H) \cdot g_3H \).
- **Identity.** The coset \( eH = H \) is the identity, since \( eH \cdot gH = (eg)H = gH \), and \( gH \cdot eH = (ge)H = gH \).
- **Inverses.** The inverse of \( gH \) is \( g^{-1}H \), since \( gH \cdot g^{-1}H = (gg^{-1})H = eH \) and \( g^{-1}H \cdot gH = (g^{-1}g)H = eH \).

Example: With \( G = A(4) \) and \( H \) as above, \( G/H = \{ H, (123)H, (124)H \} \).

Since \( |G/H| = 3 \), a prime, the group \( G/H \) must be isomorphic to \( \mathbb{Z}_3 \).
In fact the composition table for $G/H$ is

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**Exercises.**

(1) “A completely analogous argument proves that, since $H$ is a subgroup, the relation $\equiv_H$ is also an equivalence relation.” Write out the details of this argument.

(2) In $G = \mathbb{Z}_{21}^*$ show that the set $H = \{1, 4, 16\}$ is a subgroup. Since $G$ is abelian, $H$ is automatically normal. Identify the four elements of $G/H$ and construct their multiplication table. Is this group isomorphic to $\mathbb{Z}_4$?

(3) Let $H$ be a subgroup of $G$. Show that $H$ is normal if and only if $ghg^{-1} \in H$ for every $g \in G, h \in H$. Another way of writing this is: $gHg^{-1} = H$ for every $g \in G$. 