MAT 312/AMS 351
Notes and Exercises on Permutations and Matrices.

We can represent a permutation \( \pi \in S(n) \) by a matrix \( M_\pi \) in the following useful way. If \( \pi(i) = j \), then \( M_\pi \) has a 1 in column \( i \) and row \( j \); the entries are 0 otherwise. This \( M_\pi \) permutes the unit column vectors \( e_1, e_2, \ldots, e_n \), by matrix multiplication, just the way \( \pi \) permutes 1, 2, \ldots, \( n \).

**Example.** Suppose \( n = 6 \) and \( \pi = (1542)(36) \). Following the rule, we get

\[
M_\pi = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

We can check: \( \pi(1) = 5 \), and

\[
M_\pi(e_1) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix}1 \end{pmatrix} = \begin{pmatrix}0 \\
0 \\
0 \\
0 \\
0 \\
1 \\
0 \end{pmatrix} = e_5,
\]

etc.

**Proposition 1.** For \( \sigma, \pi \in S(n) \), we have \( M_{\pi\sigma} = M_\pi M_\sigma \); i.e. the matrix corresponding to a composition of permutations is the product of the individual matrices.

**Proof.** On the one hand, \( M_{\pi\sigma}(e_i) = e_{\pi\sigma(i)} = e_{\pi(\sigma(i))} \). On the other hand, \( M_\pi M_\sigma(e_i) = M_\pi(e_{\sigma(i)}) = e_{\pi(\sigma(i))} \) also. \( \square \)

To proceed we need some facts about determinants.

(1) Every square matrix \( M \) has a determinant \( \det M \), which is a sum of products of entries in \( M \). So if \( M \) has integer entries, \( \det M \) will be an integer, etc.

(2) The determinant of a \( 1 \times 1 \) matrix \( (a_{11}) \) is the number \( a_{11} \) itself. The determinant of the \( 2 \times 2 \) matrix

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22} \\
\end{pmatrix}
\]

is

\[
a_{11}a_{22} - a_{12}a_{21}.
\]
is $a_{11}a_{22} - a_{12}a_{21}$ and working by induction the determinant of the $n \times n$ matrix

$$
M = \begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
a_{21} & \cdots & a_{2n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{pmatrix}
$$

is $\det M = a_{11} \det M_{11} - a_{12} \det M_{12} + \cdots \pm a_{1n} \det M_{1n}$ where the signs alternate, and $M_{1k}$ is the matrix obtained from $M$ by striking out the first row and the $k$th column.

(3) If $I$ is the $n \times n$ identity matrix

$$
I = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}
$$

then $\det I = 1$. This follows from the construction above.

(4) If matrix $M'$ is obtained from matrix $M$ by permuting two rows, then $\det M' = -\det M$. (This is also true for columns, but we’ll be working with rows).

(5) $\det(MN) = \det M \det N$. These last two facts are not obvious. Consult any Linear Algebra text for proofs.

**Proposition 2.** If $M_\pi$ is the matrix corresponding to a permutation $\pi$, then $\det M = \pm 1$.

**Proof.** There is exactly one row of $M$ with a 1 in the first column. If it is not already at the top, it can be switched with the top row. Similarly the unique row with a 1 in column 2 can be placed in second position, etc. Each time the determinant changes by a factor of $-1$ (if the row has moved) or 1 if it stays the same. At the end we have an identity matrix (with determinant 1); and a sign which is the product of all the $-1$s accumulated during the process. □

*Shorter proof.* Write $\pi^{-1}$ for the inverse permutation. Then since $\pi^{-1}\pi = e$ (the identity permutation) Prop. 1 tells us that $M_{\pi^{-1}}M_\pi = I$ (the identity matrix). So by Fact 5, $\det M_{\pi^{-1}} \det M_\pi = 1$. Since $\det M_\pi$ divides 1, it must equal 1 or -1. □
Definition: The sign of a permutation \( \pi \in S(n) \) is defined to be the determinant of the corresponding matrix:

\[ \operatorname{sgn} \pi = \det M_\pi. \]

Proposition 3. Write \( \pi \) as a product of transpositions (permutations that exchange 2 elements and leave the others fixed; this can be done in many different ways). Then

\[ \operatorname{sgn} \pi = (-1)^{\text{number of transpositions}}. \]

Proof. Suppose the transpositions are \( \tau_1, \tau_2, \ldots, \tau_N \) so that
\[ \pi = \tau_N \tau_{N-1} \cdots \tau_2 \tau_1. \]

Then by repeated application of Prop. 1,
\[ M_\pi = M_{\tau_N} M_{\tau_{N-1}} \cdots M_{\tau_2} M_{\tau_1}, \]

and by repeated application of Fact 5 above,
\[ \det M_\pi = \det M_{\tau_N} \det M_{\tau_{N-1}} \cdots \det M_{\tau_2} \det M_{\tau_1}. \]

Now since a transposition \( \tau = (ij) \) exchanges elements \( i \) and \( j \) and leaves the others fixed, the matrix \( M_\tau \) must have the form

\[
\begin{bmatrix}
1 & \cdots & i & \cdots & j & \cdots & n \\
1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 1 
\end{bmatrix}
\]

with 1s along the diagonal except in rows \( i \) and \( j \). Since a single row swap makes this the identity matrix, we have \( \operatorname{sgn} \tau = -1 \). Since this holds for every transposition, we have \( \operatorname{sgn} \pi = (-1)^N \), as desired. \( \square \)
Exercises.

(1) Working in $S(3)$, write down the matrices corresponding to $\pi = (123)$ and to $\sigma = (12)$. Calculate the matrix products $M_\pi M_\sigma$ and $M_\sigma M_\pi$. Check that these correspond to the permutations $\pi \sigma = (13)$ and $\sigma \pi = (23)$.

(2) Working in $S(6)$, write the permutation $(1346)(25)$ as a product of transpositions in two different ways and with different numbers of transpositions. Please do not use copies $(ij)(ij)$ of a transposition and its inverse to pad your lists.

(3) What is the sign of the permutation that takes a list of $n$ things and writes it in reverse order?

(4) Show that a permutation and its inverse have the same sign.

(5) The Alternating Group $A(4)$ consists of the 12 even permutations of 4 elements. Make a list of the 12, in cycle notation. Explain why in general $A(n)$ is closed under composition (i.e. why if $\sigma \in A(n)$, $\pi \in A(n)$ then $\sigma \pi \in A(n)$), and why if $\pi \in A(n)$ then $\pi^{-1} \in A(n)$. This makes $A(n)$ a subgroup of $S(n)$.

(6) Still working with $A(n)$, explain why if $\pi \in A(n)$ and $\sigma$ is any permutation in $S(n)$, then $\sigma \pi \sigma^{-1} \in A(n)$.

(7) Prove that for any nonempty subset $H$ of a group $G$ with composition law $\ast$, the condition
   - If $h, k \in H$ then $h \ast k^{-1} \in H$.

is equivalent to the two conditions
   - If $h, k \in H$ then $h \ast k \in H$.
   - If $h \in H$ then $h^{-1} \in H$. 