MAT 312/AMS 351
Notes and Exercises on Permutations and Matrices.

We can represent a permutation $\pi \in S(n)$ by a matrix M_π in the following useful way. If $\pi(i) = j$, then M_π has a 1 in column i and row j; the entries are 0 otherwise. This M_π permutes the unit column vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, by matrix multiplication, just the way π permutes $1, 2, \ldots, n$.

Example. Suppose $n = 6$ and $\pi = (1542)(36)$. Following the rule, we get

$$M_\pi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We can check: $\pi(1) = 5$, and

$$M_\pi(\mathbf{e}_1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \mathbf{e}_5,$$

etc.

Proposition 1. For $\sigma, \pi \in S(n)$, we have $M_{\pi\sigma} = M_\pi M_\sigma$; i.e. the matrix corresponding to a composition of permutations is the product of the individual matrices.

Proof. On the one hand, $M_{\pi\sigma}(\mathbf{e}_i) = \mathbf{e}_{\pi\sigma(i)} = \mathbf{e}_{\pi(\sigma(i))}$. On the other hand, $M_\pi M_\sigma(\mathbf{e}_i) = M_\pi(\mathbf{e}_{\sigma(i)}) = \mathbf{e}_{\pi(\sigma(i))}$ also. □

To proceed we need some facts about determinants.

(1) Every square matrix M has a determinant $\det M$, which is a sum of products of entries in M. So if M has integer entries, $\det M$ will be an integer, etc.

(2) The determinant of a 1×1 matrix (a_{11}) is the number a_{11} itself. The determinant of the 2×2 matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is $a_{11}a_{22} - a_{12}a_{21}$ and working by induction the determinant of the $n \times n$ matrix

$$M = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

is $\det M = a_{11} \det M_{11} - a_{12} \det M_{12} + \cdots \pm a_{1n} \det M_{1n}$ where the signs alternate, and M_{1k} is the matrix obtained from M by striking out the first row and the kth column.

(3) If I is the $n \times n$ identity matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

then $\det I = 1$. This follows from the construction above.

(4) If matrix M' is obtained from matrix M by permuting two rows, then $\det M' = -\det M$. (This is also true for columns, but we'll be working with rows).

(5) $\det(MN) = \det M \det N$. These last two facts are not obvious. Consult any Linear Algebra text for proofs.

Proposition 2. If M_π is the matrix corresponding to a permutation π, then $\det M = \pm 1$.

Proof. There is exactly one row of M with a 1 in the first column. If it is not already at the top, it can be switched with the top row. Similarly the unique row with a 1 in column 2 can be placed in second position, etc. Each time the determinant changes by a factor of -1 (if the row has moved) or 1 if it stays the same. At the end we have an identity matrix (with determinant 1); and a sign which is the product of all the -1s accumulated during the process. □

Shorter proof. Write π^{-1} for the inverse permutation. Then since $\pi^{-1}\pi = e$ (the identity permutation) Prop. 1 tells us that $M_{\pi^{-1}} M_\pi = I$ (the identity matrix). So by Fact 5, $\det M_{\pi^{-1}} \det M_\pi = 1$. Since $\det M_\pi$ divides 1, it must equal 1 or -1. □

Definition: The sign of a permutation $\pi \in S(n)$ is defined to be the determinant of the corresponding matrix:

$$\text{sgn } \pi = \det M_\pi.$$

Proposition 3. Write π as a product of transpositions (permutations that exchange 2 elements and leave the others fixed; this can be done in many different ways). Then

$$\text{sgn } \pi = (-1)^{\text{number of transpositions}}.$$

Proof. Suppose the transpositions are $\tau_1, \tau_2, \ldots, \tau_N$ so that

$$\pi = \tau_N \tau_{N-1} \cdots \tau_2 \tau_1.$$

Then by repeated application of Prop. 1,

$$M_\pi = M_{\tau_N} M_{\tau_{N-1}} \cdots M_{\tau_2} M_{\tau_1}$$

and by repeated application of Fact 5 above,

$$\det M_\pi = \det M_{\tau_N} \det M_{\tau_{N-1}} \cdots \det M_{\tau_2} \det M_{\tau_1}.$$

Now since a transposition $\tau = (ij)$ exchanges elements i and j and leaves the others fixed, the matrix M_τ must have the form

$M_\tau =$

	1	...	i	...	j	...	n
1	1	...	0	...	0	...	0
...	...	...	...	...	...	...	...
i	0	...	0	...	1	...	0
...	...	...	...	...	...	...	...
j	0	...	1	...	0	...	0
...	...	...	...	...	...	...	...
n	0	...	0	...	0	...	1

with 1s along the diagonal except in rows i and j. Since a single row swap makes this the identity matrix, we have sgn $\tau = -1$. Since this holds for every transposition, we have sgn $\pi = (-1)^N$, as desired. □

Exercises.

(1) Working in $S(3)$, write down the matrices corresponding to $\pi = (123)$ and to $\sigma = (12)$. Calculate the matrix products $M_\pi M_\sigma$ and $M_\sigma M_\pi$. Check that these correspond to the permutations $\pi\sigma = (13)$ and $\sigma\pi = (23)$.

(2) Working in $S(6)$, write the permutation $(1346)(25)$ as a product of transpositions in two different ways and with different numbers of transpositions. Please do not use copies $(ij)(ij)$ of a transposition and its inverse to pad your lists.

(3) What is the sign of the permutation that takes a list of n things and writes it in reverse order?

(4) Show that a permutation and its inverse have the same sign.

(5) The Alternating Group $A(4)$ consists of the 12 *even* permutations of 4 elements. Make a list of the 12, in cycle notation. Explain why in general $A(n)$ is closed under composition (i.e. why if $\sigma \in A(n), \pi \in A(n)$ then $\sigma\pi \in A(n)$), and why if $\pi \in A(n)$ then $\pi^{-1} \in A(n)$. This makes $A(n)$ a subgroup of $S(n)$.

(6) Still working with $A(n)$, explain why if $\pi \in A(n)$ and σ is *any* permutation in $S(n)$, then $\sigma\pi\sigma^{-1} \in A(n)$.

(7) Prove that for any nonempty subset H of a group G with composition law $*$, the condition
   - If $h, k \in H$ then $h * k^{-1} \in H$.

   is equivalent to the two conditions
   - If $h, k \in H$ then $h * k \in H$.
   - If $h \in H$ then $h^{-1} \in H$.