**MAT 312/AMS 351 – Fall 2010**
**Homework 3**

1. Prove (by induction, or otherwise) that

$$(a-b)|a^n - b^n$$

for any integers $a > b$. A proof by induction would start with $a^2 - b^2 = (a-b)(a+b)$, note that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$, use this form to conjecture the general statement and then prove it.

2. Use Exercise 1 to show that if $2^n - 1$ is prime, then $n$ must be prime. [The converse is *not* true in general, but a prime of the form $2^n - 1$ is called a *Mersenne prime*. The largest prime known $2^{43112609} - 1$, discovered in 2008, is a Mersenne prime. Writing it out would require 12978189 decimal digits.]

3. What would happen to the Fundamental Theorem of Algebra if 1 were allowed to be a prime number?

4. Find the first positive integer value of $n$ such that the formula $n^2+n+29$ does not result in a prime number.

5. Consider an integer $c$ with prime factorization

$$c = p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k},$$

with $p_1, \ldots p_k$ distinct. Show that in the prime factorization of $c^n$, all the exponents are divisible by $n$.

6. Use the last execise to show that if an $n$-th power is the product of two relatively prime factors:

$$c^n = ab, \qquad (a,b) = 1$$

then each of the factors is itself an $n$-th power.