Math 310: Homework 2

Ex 1. Prove that if the list (v_1, v_2, v_3) spans V then so does the list $(v_1 + 2v_2, v_2 - v_3, v_3)$.

Ex 2. Prove that if the list (v_1, v_2, v_3) is linearly independent in V then so is the list $(v_1 + 2v_2, v_2 - v_3, v_3)$.

Ex 3. Find a basis for the vector space

$$V = \{(x_1, \dots, x_4) \in \mathbb{F}^4 : x_1 + 2x_2 + 3x_3 + 4x_4 = 0\}.$$

What is the dimension of V?

Ex 4. Suppose that $(v_1, \dots, v_n)$ is linearly independent in V.

(i) Suppose that for some $w \in V$ the list $(v_1 - w, v_2 - w, \dots, v_n - w)$ is linearly dependent. Show that $w \in \text{span}(v_1, \dots, v_n)$.

(ii) Is the converse true? That is, if $w \neq 0$ is in $\text{span}(v_1, \dots, v_n)$ must it be true that the list $(v_1 - w, v_2 - w, \dots, v_n - w)$ is linearly dependent?
Hint: What does this say when $n = 1, 2$? Try these cases first.

Ex 5. Let $\mathcal{P}(\mathbb{F})$ be the space of polynomials with coefficients in $\mathbb{F}$.

(i) Find two different 2-dimensional subspaces of $\mathcal{P}(\mathbb{F})$.

(ii) Find an infinite dimensional *proper* subspace of $\mathcal{P}(\mathbb{F})$ (i.e. a subspace that does not equal the whole of $\mathcal{P}(\mathbb{F})$.)

Ex 6. (i) Let U, V be subspaces of $\mathbb{F}^7$ such that $U \oplus V = \mathbb{F}^7$. If $\dim U = 3$ show that $\dim V = 4$.

(ii) Does this statement remain true if all you know is that $U + V = \mathbb{F}^7$? Give a proof or counterexample.

Note: In this question you may use all the results numbered up to and including 2.12. Anything else should be proved. Try to find the most econimical argument that you can.

Ex 7. In the vector space of continuous, real-valued functions, show that

$$(1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx)$$

is a linearly independent list for any integer n.

Hint. Multiply by $\cos kx$ and integrate over $[0, 2\pi]$, using the identities $2\cos ix \cos kx = \cos(i+k)x + \cos(i-k)x$ and $2\sin ix \cos kx = \sin(i+k)x - \sin(i-k)x$.