1. (i) Calculate (with answers in the form $z = a + ib$)

$$(1 + i)^2, \ (1 + i)^4.$$  

Draw a diagram of these points on the plane.

(ii) Find $z = a + ib$ with $a, b > 0$ such that $z^8 = 1$.

In the following exercises, let $V$ be a vector space over $F$ (where $F = \mathbb{R}$ or $\mathbb{C}$). You may use any proposition from Chapter 1 provided that you say explicitly where it is used.

2. (i) Let $v, w \in V$ be such that $v + w = v$. Show that $w = 0$.

(ii) Let $v \in V$ and $a \in F$ be such that $av = 0$. Show that one of $a$ or $v$ must be zero.

(iii) Let $v \in V$ and $a \in F$ be such that $av = v$. Show that one of $a = 1$ or $v = 0$.

3. For each of the following subsets $U$ of $\mathbb{R}^3$, determine whether it is a subspace of $\mathbb{R}^3$. If $U$ is a subspace, find $W$ such that $U \oplus W = \mathbb{R}^3$. Explain your answer carefully.

(a) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - 2x_2 + x_3 = 1\}$.

(b) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1)^2 - x_2 + x_3 = 0\}$.

(c) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - x_2 = 3x_3\}$.

4. Give an example of a subset $U$ of $\mathbb{R}^2$ that is closed under addition and taking additive inverses, but is not a vector space over $\mathbb{R}$.

5. Let $P_2(\mathbb{R})$ be the space of polynomials in $x$ of degree at most 2 with real coefficients. Thus $P_2(\mathbb{R}) = \{a + bx + cx^2 : a, b, c, \in \mathbb{R}\}$.

(i) Give an example of a subset $U$ of $P_2(\mathbb{R})$ that is closed under multiplication by scalars but is not a subspace.

(ii) Give an example of a subspace $U$ of $P_2(\mathbb{R})$ that is proper, i.e. not equal to $\{0\}$ or to the whole space $P_2(\mathbb{R})$.

(iii) For the subspace $U$ you found in (ii), identify another subspace $W$ such that $P_2(\mathbb{R}) = U \oplus W$.

6. Are there subspaces $U_1, U_2, W$ of $\mathbb{R}^2$ such that $U_1 \oplus W = U_2 \oplus W$ but $U_1 \neq U_2$? Give an example or prove that no such subspaces exist.

7. (Bonus problem) (i) Suppose that $U_1, U_2, U_3$ are subspaces of $V$ such that $V = U_1 + U_2 + U_3$. Formulate a condition in terms of intersections of subspaces that is equivalent to the condition that $V = U_1 \oplus U_2 \oplus U_3$.

(ii) The same question for $k$-fold sums.