MAT 310 Spring 2008 Homework 1

1. (i) Calculate (with answers in the form $z = a + ib$)

$$(1 + i)^2, \quad (1 + i)^4.$$

   Draw a diagram of these points on the plane.

   (ii) Find $z = a + ib$ with $a, b > 0$ such that $z^8 = 1$.

   In the following exercises, let V be a vector space over $\mathbf{F}$ (where $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$.) You may use any proposition from Chapter 1 provided that you say explicitly where it is used.

2. (i) Let $v, w \in V$ be such that $v + w = v$. Show that $w = 0$.

   (ii) Let $v \in V$ and $a \in \mathbf{F}$ be such that $av = 0$. Show that one of a or v must be zero.

   (iii) Let $v \in V$ and $a \in \mathbf{F}$ be such that $av = v$. Show that one of $a = 1$ or $v = 0$.

3. For each of the following subsets U of $\mathbf{R}^3$, determine whether it is a subspace of $\mathbf{R}^3$. If U is a subspace, find W such that $U \oplus W = \mathbf{R}^3$. Explain your answer carefully.

   (a) $\{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1 - 2x_2 + x_3 = 1\}$.

   (b) $\{(x_1, x_2, x_3) \in \mathbf{R}^3 : (x_1)^2 - x_2 + x_3 = 0\}$.

   (c) $\{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1 - x_2 = 3x_3\}$.

4. Give an example of a subset U of $\mathbf{R}^2$ that is closed under addition and taking additive inverses, but is not a vector space over $\mathbf{R}$.

5. Let $\mathcal{P}_2(\mathbf{R})$ be the space of polynomials in x of degree at most 2 with real cofficients. Thus $\mathcal{P}_2(\mathbf{R}) = \{a + bx + cx^2 : a, b, c, \in \mathbf{R}\}$.

   (i) Give an example of a subset U of $\mathcal{P}_2(\mathbf{R})$ that is closed under multiplication by scalars but is not a subspace.

   (ii) Give an example of a subspace U of $\mathcal{P}_2(\mathbf{R})$ that is *proper*, i.e. not equal to $\{0\}$ or to the whole space $\mathcal{P}_2(\mathbf{R})$.

   (iii) For the subspace U you found in (ii), identify another subspace W such that $\mathcal{P}_2(\mathbf{R}) = U \oplus W$.

6. Are there subspaces U_1, U_2, W of $\mathbf{R}^2$ such that $U_1 \oplus W = U_2 \oplus W$ but $U_1 \neq U_2$? Give an example or prove that no such subspaces exist.

7. (Bonus problem) (i) Suppose that U_1, U_2, U_3 are subspaces of V such that $V = U_1 + U_2 + U_3$. Formulate a condition in terms of intersections of subspaces that is equivalent to the condition that $V = U_1 \oplus U_2 \oplus U_3$.

   (ii) The same question for k-fold sums.