1. Write a complete, clear and correct proof of the following statement:
Given \( v_1 \) and \( v_2 \) eigenvectors of a linear operator; if their associated eigenvalues are different, then \( v_1 \) and \( v_2 \) are linearly independent.

**Solution.** (This is an abridged version of a proof in the book). Let \( \lambda_1 \neq \lambda_2 \) be the corresponding eigenvalues. If \( T \) is the operator, then \( Tv_1 = \lambda_1 v_1 \) and \( Tv_2 = \lambda_2 v_2 \). Suppose a linear relation \( a_1 v_1 + a_2 v_2 = 0 \). Since \( v_1 \) and \( v_2 \), being eigenvectors, are not zero, both \( a_1 \) and \( a_2 \) are nonzero. Then for example \( v_1 = -\frac{a_2}{a_1} v_2 \). so \( Tv_1 = -\frac{a_2}{a_1} \lambda_2 v_2 \); on the other hand, \( Tv_1 = \lambda_1 v_1 = -\frac{a_2}{a_1} \lambda_1 v_2 \). Since everything else is nonzero, \( \lambda_1 \) must equal \( \lambda_2 \), a contradiction. So there is no such linear relation, and \( v_1, v_2 \) are linearly independent.

2. Consider the linear operator \( T \) defined in the standard basis \((1, 0), (0, 1)\) by the matrix
\[
\begin{bmatrix}
11 & -4 \\
30 & -11
\end{bmatrix}.
\]
Take \( v = (1, 0) \) and note that \( T(v) = (11, 30) \) and \( T^2(v) = (1, 0) \), so \( T \) satisfies the equation \( (T^2 - I)v = 0 \). Factor \( T^2 - I \) as \( (T - aI)(T - bI) \). When you have calculated \( a \) and \( b \), apply \( (T - bI) \) and then \( (T - aI) \) to \( v \) to obtain an eigenvector for \( T \), as in the proof in the text that every operator on a complex vector space has an eigenvector.

**Solution.** The polynomial \( x^2 - 1 \) factors as \( (x + 1)(x - 1) \), so \( T^2 - I = (T + I)(T - I) \), giving \( a = -1, b = 1 \) in the statement of the problem. If we apply
\[
T - I = \begin{bmatrix}
10 & -4 \\
30 & -12
\end{bmatrix}
\]
to \( v = (1, 0) \) we get \((10,30)\). This is an eigenvector, with eigenvalue \(-1\) as can be checked. (If we applied \( T + I = \begin{bmatrix}
12 & -4 \\
30 & -10
\end{bmatrix}\) instead, we would get \((12,30)\), an eigenvector with eigenvalue \(1\).) Similar analysis for the other form of this problem, where \( v = (0, 1) \).
3. Give an example of a linear operator on $\mathbf{R}^2$ which is not diagonalizable but can be put in upper-triangular form.

**Solution.** Suppose $T$ is an operator which with respect to some basis is in upper-triangular form, with matrix

$$
\begin{bmatrix}
a & c \\
0 & b
\end{bmatrix}.
$$

We know that the diagonal entries are eigenvalues of $T$, and that if they are distinct the corresponding eigenvectors are linearly independent; using those eigenvectors as basis gives a diagonal matrix. So $a$ and $b$ must be equal. If $c = 0$ then our matrix is diagonal. So take for example

$$
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.
$$

We check this matrix is not diagonalizable. The only eigenvalue is 1; a possible eigenvector must satisfy

$$
A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.
$$

The first components give $x + y = x$ so $y = 0$. So any eigenvector is a multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. There cannot be two linearly independent vectors, so this transformation is not diagonalizable.