1. Write a complete, clear and correct proof of the following statement: If a list \((v_1, v_2, \ldots, v_n)\) of elements of a vector space is linearly dependent, and \(v_1 \neq 0\), then one of them can be written as a linear combination of those preceding it in the list. (I.e. for some \(j, 1 \leq j \leq n\), there are field elements \(c_1, \ldots, c_{j-1}\) such that \(v_j = c_1v_1 + c_2v_2 + \cdots + c_{j-1}v_{j-1}\)).

Solution. This was straight out of the book.

2. Given vector spaces \(V, W\) with \(\dim V = 7\) and \(\dim W = 6\). Prove that there is no injective map \(T : V \to W\).

Solution. Let \((v_1, \ldots, v_7)\) be a basis for \(V\). If \(T\) is injective, the vectors \(Tv_1, Tv_2, \ldots, Tv_7\) must be linearly independent (because if \(a_1Tv_1 + \cdots + a_7Tv_7 = 0\), with not all \(a_i = 0\), then \(v = a_1v_1 + \cdots + a_7v_7 \neq 0\), but \(Tv = 0\), contradicting injectivity). But \(W\) has a 6-element spanning set, so it cannot contain a 7-element linearly independent list.

3. Note that there were several different forms of this problem.

Let \(T : \mathbb{R}^4 \to \mathbb{R}^4\) be defined by

\[
T(x_1, x_2, x_3, x_4) = (2x_2-x_3-x_4, 2x_1+x_3+x_4, x_1-x_2+2x_3+x_4, x_1+x_2-x_3).
\]

Give a basis for the range of \(T\) and a basis for the null-space of \(T\).

Solution. Use the standard basis \(e_1 = (1, 0, 0, 0), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0), e_4 = (0, 0, 0, 1)\), and note that \(Te_1, Te_2, Te_3, Te_4\) span the range of \(T\) (since \(w \in \text{range}(T) \iff w = Tv\) for some \(v = a_1e_1 + \cdots + a_4e_4 \in V\), which makes \(w = T(a_1e_1 + \cdots + a_4e_4) = a_1Te_1 + \cdots + a_4Te_4\). So to get a basis for the range of \(T\) we go through the list \(Te_1, Te_2, Te_3, Te_4\) discarding any vector which is a linear combination of the preceding ones.

Now \(Te_1 = (0, 2, 1, 1), Te_2 = (2, 0, -1, 1), Te_3 = (-1, 1, 2, -1), Te_4 = (-1, 1, 1, 0)\). We note that \(Te_2\) cannot be a multiple of \(Te_1\), since it has a non-zero first component. Is \(Te_3\) a linear combination of \(Te_1\) and \(Te_2\)? Try

\[
(-1, 1, 2, -1) = a(0, 2, 1, 1) + b(2, 0, -1, 1).
\]
This gives four equations: \(-1 = 2b, 1 = 2a, 2 = a - b, -1 = a + b\). The first says \(b = -\frac{1}{2}\), the second says \(a = \frac{1}{2}\) but then \(a + b = 0\), contradicting equation 4. So there are no \(a\) and \(b\) that work, and \(Te_1, Te_2\) and \(Te_3\) are linearly independent. Now try the same thing with \(Te_1, Te_2, Te_3\) and \(Te_4\): write

\[
(-1, 1, 1, 0) = a(0, 2, 1, 1) + b(2, 0, -1, 1) + c(-1, 1, 2, -1).
\]

The four equations are now: \(-1 = 2b - c, 1 = 2a + c, 1 = a - b + 2c, 0 = a + b - c\). Adding equations 1 and 2 gives \(2a + 2b = 0\) or \(b = -a\). Then equation 4 gives \(c = 0\), equation 1 gives \(b = -\frac{1}{2}\), equation 2 gives \(a = \frac{1}{2}\) and equation 3 checks out \(1 = 1\). So \((-1, 1, 1, 0) = \frac{1}{2}(0, 2, 1, 1) - \frac{1}{2}(2, 0, -1, 1)\), and \(Te_4\) is not independent of the preceding vectors. So \(Te_1, Te_2\) and \(Te_3\) are a basis for the range of \(T\).

We now know that the null space of \(T\) is 1-dimensional, so it is enough to find one nonzero \(v\) with \(Tv = 0\). This means finding a non-zero solution of

\[
\begin{align*}
2x_2 - x_3 - x_4 &= 0 \quad (1) \\
2x_1 + x_3 + x_4 &= 0 \quad (2) \\
x_1 - x_2 + 2x_3 + x_4 &= 0 \quad (3) \\
x_1 + x_2 - x_3 &= 0 \quad (4).
\end{align*}
\]

Adding (1) to (2) gives \(2x_1 + 2x_2 = 0\) so \(x_1 = -x_2\). Then (4) gives \(x_3 = 0\). We can choose \(x_1\) arbitrarily, say \(x_1 = 1\). Then \(x_2 = -1, x_3 = 0\) and \(x_4 = 2x_2\) by (1) = \(-2\). A 1-vector basis of the null-space of \(T\) is \((1, -1, 0, -2)\).