MAT 310 Linear Algebra Midterm 1 Solutions

1. Write a complete, clear and correct proof of the following statement: If a list $(v_1, v_2, \ldots, v_n)$ of elements of a vector space is linearly dependent, and $v_1 \neq 0$, then one of them can be written as a linear combination of those preceding it in the list. (I.e. for some j, $1 \leq j \leq n$, there are field elements $c_1, \ldots, c_{j-1}$ such that $v_j = c_1v_1 + c_2v_2 + \cdots + c_{j-1}v_{j-1}$).

   Solution. This was straight out of the book.

2. Given vector spaces V, W with $\dim V = 7$ and $\dim W = 6$. Prove that there is no injective map $T : V \to W$.

   Solution. Let $(v_1, \ldots, v_7)$ be a basis for V. If T is injective, the vectors $Tv_1, Tv_2, \ldots, Tv_7$ must be linearly independent (because if $a_1Tv_1+\cdots+a_7Tv_7 = 0$, with not all $a_i = 0$, then $v = a_1v_1 + \cdots + a_7v_7$ is $\neq 0$, but $Tv = 0$, contradicting injectivity). But W has a 6-element spanning set, so it cannot contain a 7-element linearly independent list.

3. *Note that there were several different forms of this problem.*

   Let $T : \mathbf{R}^4 \to \mathbf{R}^4$ be defined by

   $$T(x_1, x_2, x_3, x_4) = (2x_2-x_3-x_4, 2x_1+x_3+x_4, x_1-x_2+2x_3+x_4, x_1+x_2-x_3).$$

   Give a basis for the range of T and a basis for the null-space of T.

   Solution. Use the standard basis $e_1 = (1, 0, 0, 0), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0), e_4 = (0, 0, 0, 1)$, and note that Te_1, Te_2, Te_3, Te_4 span the range of T (since $w \in \text{range}(T) \iff w = Tv$ for some $v = a_1e_1 + \ldots + a_4e_4 \in V$, which makes $w = T(a_1e_1 + \ldots + a_4e_4) = a_1Te_1 + \ldots + a_4Te_4$. So to get a basis for the range of T we go through the list Te_1, Te_2, Te_3, Te_4 discarding any vector which is a linear combination of the preceding ones.

   Now $Te_1 = (0, 2, 1, 1), Te_2 = (2, 0, -1, 1), Te_3 = (-1, 1, 2, -1), Te_4 = (-1, 1, 1, 0)$. We note that Te_2 cannot be a multiple of Te_1, since it has a non-zero first component. Is Te_3 a linear combination of Te_1 and Te_2? Try

   $$(-1, 1, 2, -1) = a(0, 2, 1, 1) + b(2, 0, -1, 1).$$

This gives four equations: $-1 = 2b, 1 = 2a, 2 = a - b, -1 = a + b$. the first says $b = -\frac{1}{2}$, the second says $a = \frac{1}{2}$ but then $a + b = 0$, contradicting equation 4. So there are no a and b that work, and Te_1, Te_2 and Te_3 are linearly independent. Now try the same thing with Te_1, Te_2, Te_3 and Te_4: write

$$(-1, 1, 1, 0) = a(0, 2, 1, 1) + b(2, 0, -1, 1) + c(-1, 1, 2, -1).$$

The four equations are now: $-1 = 2b - c, 1 = 2a + c, 1 = a - b + 2c, 0 = a + b - c$. Adding equations 1 and 2 gives $2a + 2b = 0$ or $b = -a$. Then equation 4 gives $c = 0$, equation 1 gives $b = -\frac{1}{2}$, equation 2 gives $a = \frac{1}{2}$ and equation 3 checks out $1 = 1$. So $(-1, 1, 1, 0) = \frac{1}{2}(0, 2, 1, 1) - \frac{1}{2}(2, 0, -1, 1)$, and Te_4 is not independent of the preceding vectors. So Te_1, Te_2 and Te_3 are a basis for the range of T.

We now know that the null space of T is 1-dimensional, so it is enough to find one nonzero v with $Tv = 0$. This means finding a non-zero solution of

$$2x_2 - x_3 - x_4 = 0 \quad (1)$$

$$2x_1 + x_3 + x_4 = 0 \quad (2)$$

$$x_1 - x_2 + 2x_3 + x_4 = 0 \quad (3)$$

$$x_1 + x_2 - x_3 = 0 \quad (4).$$

Adding (1) to (2) gives $2x_1 + 2x_2 = 0$ so $x_1 = -x_2$. Then (4) gives $x_3 = 0$. We can choose x_1 arbitrarily, say $x_1 = 1$. Then $x_2 = -1$, $x_3 = 0$ and $x_4 = 2x_2$ by $(1) = -2$. A 1-vector basis of the null-space of T is $(1, -1, 0, -2)$.