2.5 Understand how change of bases affects the matrix $[T]_{\beta}$ of a linear transformation $T: V \to W$. Namely if $\beta'$ is a new basis for $V$ and $\gamma'$ a new basis for $W$ write $T$ as the composition $T = I_W \circ T \circ I_V$:

$$V \xrightarrow{I_V} V \xrightarrow{T} W \xrightarrow{I_W} W$$

going from $V$ (basis $\beta'$) to $V$ (basis $\beta$) to $W$ (basis $\gamma$) to $W$ (basis $\gamma'$), so

$$[T]_{\beta'}^{\gamma'} = [I_W]_{\gamma'}^{\gamma}[T]_{\beta}^{\gamma}[I_V]_{\beta}^{\beta'}.$$  

Also, be able to calculate $[I_V]_{\beta}^{\beta'}$: suppose $\beta = (v_1, \ldots, v_n)$ and $\beta' = (v'_1, \ldots, v'_n)$; then the first column of $[I_V]_{\beta}^{\beta'}$ is the column of coefficients obtained when $v'_1$ is written as a linear combination of $v_1, \ldots, v_n$, i.e. it’s the vector $v'_1$ written in the basis $(v_1, \ldots, v_n)$. Similarly second column of $[I_V]_{\beta}^{\beta'}$ is the vector $v'_2$ written in the basis $(v_1, \ldots, v_n)$, etc. This is especially simple when $\beta$ is the standard basis.

On the other hand if you know $[I_W]_{\gamma}^{\gamma'}$, the matrix $[I_W]_{\gamma}^{\gamma'}$ can be retrieved by inverting $[I_W]_{\gamma}^{\gamma'}$, since $[I_W]_{\gamma}^{\gamma'}[I_W]_{\gamma'}^{\gamma} = [I_W]_{\gamma}^{\gamma} = I$, the identity matrix. *Examples 1, 2; Corollary, p.115; Problem 6d*

3.1 Understand the three kinds of elementary row operations, and how each of them can be carried out on a matrix $A$ by *left*-multiplying $A$ with the appropriate *elementary matrix* (which is the matrix obtained by applying that row operation to the identity matrix!). *Theorem 3.1, Example 2*. Be able to invert an elementary matrix on inspection (be able to prove *Theorem 3.2*; pay attention to type 3.) Understand this paragraph when “row” $\to$ “column” and “left” $\to$ “right.” *Problems 2, 4, 7*

3.2 Remember that the *rank* of a linear transformation is the dimension of its range. Understand that the rank of an $m \times n$ matrix $A$ is the rank of $L_A: \mathbb{F}^n \to \mathbb{F}^m$, and *Theorem 3.3*. Understand why the rank of $A$ is
the maximum number of linearly independent columns in $A$ (*Theorem 3.5, Examples 1, 2*). Understand the content of *Theorem 3.6*: After an appropriate change of basis, an arbitrary linear $T:F^n \rightarrow F^m$ becomes $(a_1, a_2, \ldots, a_r, a_{r+1}, \ldots, a_n) \rightarrow (a_1, a_2, \ldots, a_r, 0, \ldots, 0)$ where $r$ is the rank of $T$. Understand how this theorem and its proof imply *Corollary 2*: $\text{rank}(A^t) = \text{rank}(A)$ and *Corollary 3*: Every invertible matrix is a product of elementary matrices which are both very useful.

Know how to compute the inverse of an invertible matrix $A$ by forming the augmented matrix $A;I$ and row-reducing it to get $I,A^{-1}$ *Examples 5, 6*: if $A$ cannot be row-reduced to $I$, then $A$ is not invertible.

4.1 Understand how to compute the determinant of a $2 \times 2$ matrix $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$, and the connection between determinants and area: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$ the area of the parallelogram spanned by the vectors $(a, b)$ and $(c, d)$ with a plus sign if $((a, b), (c, d))$ is a right-handed system and minus otherwise. *Problems 2, 3, 4.*

4.2 Understand how to carry out the inductive calculation of the determinant of a large square matrix, and be able to do it for $3 \times 3$ and $4 \times 4$ matrices. *Examples 1, 2, 3*. Understand the proof of *Theorem 4.3*: the function $A \rightarrow \det A$ is linear in each row separately. Understand the statement of the Lemma on pp. 213-214, and how it and *Theorem 4.3* imply *Theorem 4.4*: determinant can be calculated by cofactor expansion along any row. Understand how this implies that if two rows of $A$ are identical, then $\det A = 0$; and furthermore that if $A'$ is derived from $A$ by interchanging 2 rows, then $\det A' = -\det A$. Be able to use these concepts to prove *Theorem 4.6*: det is invariant under type-3 elementary row operations and its *Corollary* p. 217: if $A$ is $n \times n$ and $\text{rank}(A) < n$ then $\det A = 0$. Finally be able to calculate the determinant of a large matrix by using row operations to simplify the problem. *Examples 5, 6*. *Problems 9, 21.*

4.3 Know how to prove the important *Theorem 4.7*: $\det(AB) = \det A \cdot \det B$ and the related *Theorem 4.8*: $\det A^t = \det A$ ($A^t$ the transpose). *Problems 8, 9.*