MAT 200
SOLUTIONS TO HOMEWORK 6

OCTOBER 21, 2004

Section 4: 4.1, 4.2

4.1 To prove that there exists a unique point C satisfying

$$(m\angle BAC = \alpha) \wedge (\,|AC| = r) \wedge (C \text{ is in } H) \tag{1}$$

we need to prove existence and we also need to prove uniqueness.

Existence: By Protractor Axiom (3), there exists a ray $\overrightarrow{AD}$ in halfplane H such that $m\angle BAD = \alpha$. By Exercise 3.1, there exists a point C on the ray $\overrightarrow{AD}$ such that $|AC| = r$. Since $C \in \overrightarrow{AD}$, $\overrightarrow{AC} = \overrightarrow{AD}$ by Theorem 3.4. So, $m\angle BAC = m\angle BAD = \alpha$. Thus, such a point C satisfies (1).

Uniqueness: Assume that C_1, C_2 are two points satisfying (1). Since $m\angle BAC_1 = m\angle BAC_2 = \alpha$, by uniquenes statement of Protractor axiom we have $\overrightarrow{AC_1} = \overrightarrow{AC_2}$. Since $|AC_1| = |AC_2| = r$ and C_1, C_2 are on the same ray starting at A, by uniqueness statement of Exercise 3.1, $C_1 = C_2$

(In fact, one could give a shorter proof, proving existence and uniqueness together.)

4.2 The counterexample

Using the Protractor Axiom, we may choose two points C and D such that lie on different sides of $\overleftrightarrow{AB}$, and $m\angle BAD < m\angle BAC$ (for example, by taking $m\angle BAD = \pi/4, m\angle BAC = \pi/3$). But since C and D are on different sides of $\overleftrightarrow{AB}$, $\overrightarrow{AD}$ is not inside the angle $\angle BAC$ (strictly speaking, this also requires proof — but since we didn't really give an accurate definition of "inside", we omit the proof. It could be done, e.g., using the crossbar theorem).

Thus, in this example $m\angle BAD < m\angle BAC$ but $\overrightarrow{AD}$ is not inside the angle $\angle BAC$.

4.3 Denote $\alpha = m\angle BAC$. Let $\overrightarrow{AD}$ be a ray which is on the same side of $\overleftrightarrow{AB}$ as C and such that $m\angle BAD = \alpha/2$ (such a ray exists by Protractor Axiom). Then $\overrightarrow{AD}$ is inside $\angle BAC$ (Theorem 4.2) and thus, by Protractor axiom, $m\angle DAC = \alpha - \alpha/2 = \alpha/2$, so $m\angle DAC = m\angle BAD$. This shows existence.

To prove uniqueness, note that if $\overrightarrow{AD}$ is a bisector, then we must have $m\angle BAD = m\angle DAC$. Since, by protractor axiom, $m\angle BAD + m\angle DAC = m\angle BAC$, we must have $m\angle BAD = m\angle BAC/2$. Similarly, id $\overrightarrow{AD'}$ is another bisector, then similar argument gives $m\angle BAD' = m\angle BAC/2$. Then $m\angle BAD = m\angle BAD'$, so by protractor axiom, $\overrightarrow{AD} = \overrightarrow{AD'}$. This proves uiniqeness.

Section 5: 5.2, 5.3, 5.5, 5.6

5.2 The Gap:

The Protractor Axiom lets us find a point D such that $m\angle BCD = m\angle B'C'A'$, but we do not know if we can find this point on the segment AB. To do this we need to use Theorems 4.1 and 4.2.

Filling the Gap:

Apply the Protractor Axiom to find a ray $\overrightarrow{CE}$ where E lies on the same half plane as B and $m\angle BCE = m\angle B'C'A'$. Since $m\angle BCA > m\angle B'C'A' = m\angle BCE$, by Monotonicity of Angles theorem (Theorem 4.2), the ray $\overrightarrow{CE}$ is inside the angle $\angle BCA$. Now, by the Crossbar theorem (Theorem 4.1), this ray intersects the segment AB at some point, say D. By Theorem 3.4, $\overrightarrow{CE} = \overrightarrow{CD}$, so $m\angle BCD = m\angle BCE = m\angle B'C'A'$.

5.3 (1). By the definition of midpoint (Exercise 3.3), $|AM| = |MC|$ and $|DM| = |MB|$. Also by Theorem 4.3, vertical angles are equal, i.e. $m\angle AMD = m\angle CMB$. So $\triangle AMD \cong \triangle CMB$ by SAS(Theorem 5.1).
Similarly, we also have $\triangle AMB \cong \triangle CMD$.

(2), (3). Follow from (1) and definition of congruent triangles.

(4) By Protractor axiom, $m\angle ABC = m\angle ABM + m\angle MBC$, and $m\angle ADC = m\angle ADM + m\angle MDC$. Using congruences above, $m\angle ABM = m\angle MDC$, $m\angle MBC = m\angle MDA$. Adding these two equalities, we get $m\angle ABC = m\angle ADC$.

5.5 Suppose D is on $\overleftrightarrow{BC}$. By Crossbar Theorem (Theorem 4.1), D must be on the segment BC, so D is between B, C, Now $\triangle A'B'C' \cong \triangle ABD$ by SAS. ($|AB| = |A'B'|$,$|AD| = |A'C'|$, and $m\angle A' = m\angle BAD$). Thus we have $|BC| = |B'C'| = |BD|$. On the other hand, by Theorem 3.6, since D is between B, C we have $|BC| = |BD| + |DC|$. Thus, $|DC| = 0$, so $D = C$. But this contradicts to D being between B, C.

5.6 $\triangle ADC$ is isosceles because $|AD| = |A'C'| = |AC|$. Thus, by Theorem 5.2 $m\angle ADC = m\angle ACD$. Denote $\alpha = m\angle ADC = m\angle ACD$.

Similarly, $|BD| = |B'C'| = |BC|$, so $\triangle BDC$ is isosceles. Thus, by Theorem 5.2 $m\angle BDC = m\angle BCD$. Denote $\beta = m\angle BDC = m\angle BCD$.

Now, since AD crosses BC, $\overrightarrow{DA}$ is inside the angle $\angle BDC$. Using Theorem 4.2, we have $m\angle ADC < m\angle BDC$, i.e., $\alpha < \beta$.

Since AD crosses BC, $\overrightarrow{CB}$ is inside the angle $\angle ACD$. Using Theorem 4.2, we have $m\angle BCD < m\angle ACD$, i.e. $\beta < \alpha$.

Thus we have $\alpha < \beta$ and $\beta < \alpha$, which is a contradiction.