MAT 200
SOLUTIONS TO HOMEWORK 5

OCTOBER 12, 2004

Section 2: 2.1, 2.3, 2.4

In the solutions below, we use $C(l,m)$ for "l,m have common points"; formally it can be written as $\exists P\ (P \in m \wedge P \in l)$.

2.4. The given definitions are:
"l,m intersect" $\leftrightarrow (l \neq m) \wedge C(l,m)$.
$l \| m \leftrightarrow (l = m) \vee \sim C(l,m)$.
Thus, by De Morgan's law,

$$\sim (l \| m) \leftrightarrow (l \neq m) \wedge C(l,m)$$

which is the definition of "l,m intersect". Thus,

$$(l,m \text{ intersect }) \leftrightarrow \ \sim (l \| m)$$

by substitution.

2.3 We are given that $C \in \overleftrightarrow{AB}$; also, by definition, $B \in \overleftrightarrow{AB}$. But by Incidence Axiom, there is a unique line containing B, C, and we denoted this line by $\overleftrightarrow{BC}$. Thus, $\overleftrightarrow{AB} = \overleftrightarrow{BC}$. Since $A \in \overleftrightarrow{AC}$ (by definition of $\overleftrightarrow{AC}$), this implies that $A \in \overleftrightarrow{BC}$.

2.4 Proof by contradiction. Assume that n does not intersect m. By Exercise 2.1, this means $m \| n$. Using Theorem 2.2, we have $l \| n$. So again by Exercise 2.1, l and n are not intersecting. But we are given that l and n are intersecting. Thus, we have a contradiction. Thus, our assumption was false.

Theorem 3.4, 3.1–3.3

Thm. 3.4 By the Exercise 2.3, $\overleftrightarrow{VB} = \overleftrightarrow{VA}$. Since V divides the line into two non-intersecting rays, we either have $\overrightarrow{VB} = \overrightarrow{VA}$ or $\overrightarrow{VB}$ has no common points with $\overrightarrow{VA}$. But the second case cannot be true since $B \in \overrightarrow{VA}$ and $B \in \overrightarrow{VB}$.

3.1 Choose a coordinate system on $\overleftrightarrow{VA}$ such that $f(V) = 0, f(A) > 0$ (this is possible by Theorem 3.1). Then $P \in \overrightarrow{VA}$ if and only if $f(P) > 0$, and condition $|VP| = r$ is equivalent to $|f(P)| = r$. Thus, $P \in \overrightarrow{VA} \wedge |VP| = r \leftrightarrow (f(P) > 0) \wedge |f(P)| = r$. BUt the two conditions $f(P) >) \wedge |f(P)| = r$ have a unique solution (namely, $f(P) = r$), so there is a unique point P satisfying $P \in \overrightarrow{VA} \wedge |VP| = r$

3.2 Choose a coordinate system on $\overleftrightarrow{VA}$ such that $f(V) = 0,\ f(A) > 0$ (we can do it by Theorem 3.1).

Since $B \in \overrightarrow{VA}$, $f(B) > 0$ (If not, then we would have $f(B) < 0 = f(V) < f(A)$ which means V is between A and B). Also $|VB| = |f(V) - f(B)| = |f(B)| = f(B)$ and $|VA| = |f(V) - f(A)| = |f(A)| = f(A)$. So from $|VB| < |VA|$ we have $0 < f(B) < f(A)$. Hence B lies between V and A.

3.3 Choose a coordinate system on $\overleftrightarrow{AB}$ such that $f(A) = 0$, $f(B) > 0$. Then let $b = f(B)$.

To prove existence of M satisfying

$$\begin{aligned} &|AM| = |MB| \\ &M \text{ is between } A, B \end{aligned} \tag{1}$$

take M to be a point on the line $\overleftrightarrow{AB}$ such that $f(M) = b/2$. Then $0 < b/2 < b$, so M is between A, B, and $|AM| = |b/2| = b/2, |MB| = |b - b/2| = b/2$. Thus, such a point satisfies (1). This proves existence.

To prove uniqueness of a point M satisfying (1), assume that M_1, M_2 are two such points. Let $x_1 = f(M_1), x_2 = f(M_2)$. Then (1) gives

$$\begin{aligned} &|x_1| = |b - x_1| \\ &0 < x_1 < b \end{aligned}$$

which implies $x_1 = b - x_1$, so $x_1 = b/2$. Similarly (1) for M_2 gives $x_2 = b/2$. So $x_1 = x_2$, and $M_1 = M_2$. This proves uniqueness.

(A short version of the above argument: introduce a coordinate system such that $f(A) = 0, f(B) = b > 0$; then (1) is equivalent to the system

$$\begin{aligned} &x = |b - x| \\ &0 < x < b \end{aligned}$$

which has a uniqe solution $x = b/2$).