MAT 303 Lecture 9

$P(x) = a_n x^n + \dots + a_0$

$P(D) = a_n D^n + \dots + a_0$

Exponential shift theorem:

$$P(D)\,[ue^{ax}] = e^{ax}\,P(D+a)\,[u]$$

Proof: it suffices to prove this for $P(x) = x^k$

since $$(P+Q)(D)\,[ue^{ax}] = P(D)[ue^{ax}] + Q(D)[ue^{ax}]$$

Argue by induction:

k=0: $1\,[ue^{ax}] = e^{ax}\,1\,[D \mapsto D+a]\,[e^{ax}]$

inductive case: Assume that $P(x) = x^k$ that the result holds

$$D^{k+1}[ue^{ax}] = D[D^k[ue^{ax}]]$$

$$= D\,[e^{ax}(D+a)^k[u]]$$

$$= ae^{ax}(D+a)^k[u] + e^{ax}D(D+a)^k[u]$$

$$= e^{ax}(D+a)(D+a)^{n}[u] = e^{ax}(D+a)^{n+1}[u]$$

$\boxed{z}$

Example:

$$(D^2 - 2D - 3)\left[x^3 e^{-2x}\right]$$

$$= (D-3)(D+1)\left[x^3 e^{-2x}\right]$$

$$= e^{-2x}(D-5)(D-1)\left[x^3\right]$$

$$= e^{-2x}(D^2 - 6D + 5)\left[x^3\right]$$

$$= e^{-2x}\left(6x - 18x^2 + 5x^3\right) \quad \square$$

Inverse operators:

If we want to solve $P(D)[y] = f(x)$

Let's just write $y = \frac{1}{P(D)}[f]$

This just means that $\frac{1}{P(D)}$ is some operator such that $\frac{1}{P(D)}[f]$ solves the equation.

$$y = \frac{1}{D^2}[2x+3] \iff D^2 y = 2x+3$$

$$\Rightarrow y = \frac{1}{3}x^3 + \frac{3}{2}x^2 + Cx + D$$

The solutions to the homogeneous eq'n are $Cx+D$ so we will pick the simplest solution that removes any addative factors that solve the homogeneous eq'n.

So we'll say $\frac{1}{D^2}[2x+3] = \frac{1}{3}x^3 + \frac{3}{2}x^2$

$\frac{1}{D-a_0}[bx^k]$ in other words, I want to solve

$$y' - a_0 y = bx^k$$

$a_0 \neq 0$

solution without inverse operators:

ansatz: $y = A_k x^k + \dots + A_0$

Then $y' = kA_k x^{k-1} + (k-1)A_{k-1}x^{k-2} + \dots + A_1$

$$y' - a_0 y = -a_0 A_k x^k + (kA_k - a_0 A_{k-1})x^{k-1} + \dots + (A_1 - a_0 A_0)$$

got $b = -a_0 A_k \Rightarrow A_k = \frac{-b}{a_0}$

$$0 = k A_k - a_0 A_{k-1} \Rightarrow A_{k-1} = \frac{k A_k}{a_0} = \frac{-bk}{a_0^2}$$

$$\vdots$$

$$0 = A_1 - a_0 A_0$$

$$(k-\ell) A_{k-\ell} - a_0 A_{k-\ell-1} = 0$$

$$A_{k-\ell-1} = \frac{(k-\ell) A_{k-\ell}}{a_0} \qquad A_{k-\ell} = \frac{-b k^{\underline{\ell}}}{a_0^{\ell+1}}$$

$$\frac{-b}{a_0} \quad \frac{-kb}{a_0^2} \quad \frac{-k(k-1)b}{a_0^3} \quad \cdots = \frac{-b}{a_0} \frac{k^{\underline{\ell}}}{a_0^{\ell}}$$

where $k^{\underline{0}} = 1 \quad k^{\underline{n+1}} = k(k-1)^{\underline{n}}$

e.g $6^{\underline{3}} = 6 \cdot 5 \cdot 4$

$$y = \frac{-b}{a_0}\left(x^k + \frac{k}{a_0} x^{k-1} + \frac{k(k-1)}{a_0^2} x^{k-2} + \dots + \frac{k!}{a_0^k}\right)$$

second method:

$$\frac{1}{D-a_0}\left[bx^k\right] = \frac{-1}{a_0}\,\frac{1}{1-D/a_0}\left[bx^k\right] = (*)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \rightarrow \text{Taylor expansion at } 0$$

e.g. $1 + 1/2 + 1/4 + 1/8 + \dots = \frac{1}{1-1/2} = 2$

$$(*) = \frac{-1}{a_0}\left(1 + D/a_0 + D^2/a_0^2 + \dots + D^k/a_0^k\right)\left[bx^k\right]$$

$$= \frac{-b}{a_0}\left(x^k + \frac{kx^{k-1}}{a_0} + \frac{k(k-1)x^{k-2}}{a_0^2} + \dots + \frac{k!}{a_0^k}\right)$$

Suppose $P(0) \neq 0$, i.e. $P(x) = a_k x^k + \dots + a_0$ for $a_0 \neq 0$

We want the series of $\frac{1}{a_k x^k + \dots + a_0}$

this only makes sense if $a_0 \neq 0$

if $\frac{1}{a_k x^k + \dots + a_0} = \sum_{n=0}^{\infty} b_n x^n$

then $1 = \left(a_k x^k + \dots + a_0\right)\sum_{n=0}^{\infty} b_n x^n$

$$\begin{aligned}
= a_0 b_0 &+ a_0 b_1 x + a_0 b_2 x^2 + a_0 b_3 x^3 + \dots \\
&+ a_1 b_0 x + a_1 b_1 x^2 + a_1 b_2 x^3 + \dots \\
&\qquad\quad + a_2 b_0 x^2 + a_2 b_1 x^3 + \dots \\
&\qquad\qquad\qquad + a_3 b_0 x^3 + \dots \\
&\qquad\qquad\qquad\qquad + \dots
\end{aligned}$$

$$\Rightarrow \quad 1 = a_0 b_0 \quad \Rightarrow \quad b_0 = 1/a_0$$

$$0 = a_0 b_1 + a_1 b_0 \quad \Rightarrow \quad b_1 = \frac{-a_1 b_0}{a_0} = \frac{-a_1}{a_0^2}$$

$$0 = a_0 b_2 + a_1 b_1 + a_2 b_0 \quad \Rightarrow \quad b_2 = \frac{-1}{a_0}\left(a_1 b_1 + a_2 b_0\right)$$

$$0 = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 \qquad = \frac{a_1^2}{a_0^3} - \frac{a_2}{a_0^2}$$

$$4y'' - 3y' + 9y = 5x^2$$

$$P(x) = 4x^2 - 3x + 9$$

$$\frac{1}{4x^2 - 3x + 9} = \frac{1}{9} \frac{1}{1 - \frac{1}{3}x + \frac{4}{9}x^2} = (*)$$

$$\underline{1 + \tfrac{1}{3}x - \tfrac{1}{3}x^2} + \dots$$

$$1 - \tfrac{1}{3}x + \tfrac{4}{9}x^2 \;\big)\; 1$$

$$\underline{1 - \tfrac{1}{3}x + \tfrac{4}{9}x^2}$$

$$\tfrac{1}{3}x - \tfrac{4}{9}x^2$$

$$\underline{\tfrac{1}{3}x - \tfrac{1}{9}x^2 + \tfrac{4}{27}x^3}$$

$$-\tfrac{1}{3}x^2 - \tfrac{4}{27}x^3$$

$$\underline{-\tfrac{1}{3}x^2 + \dots}$$

$$0 + \dots$$

$$(*) = \frac{1}{9}\left(1 + \tfrac{1}{3}D - \tfrac{1}{3}D^2\right)\left[5x^2\right]$$

$$= \frac{1}{9}\left(5x^2 + \tfrac{10}{3}x - \tfrac{10}{3}\right) \qquad \square$$

particular solution

What if $a_0 = 0$?

$$P(x) = x^m\left(a_n x^n + \dots + a_0\right)$$

$$P(D)[y] = 6x^k$$

$$y = \frac{1}{D^m}\left[\frac{1}{a_n x^n + \dots + a_0}[6x^k]\right] \checkmark$$

$$\text{“}=\text{”} \frac{1}{a_n x^n + \dots + a_0}\left[\frac{1}{D^m}[6x^k]\right] \rightarrow$$

this is worse

$$y'' - 2y' = 5$$

$$y = \frac{1}{D}\left[\frac{1}{D-2}[5]\right] = \frac{1}{D}\left[\frac{1}{-2}\,\frac{1}{1-D/2}[5]\right]$$

$$= \frac{1}{D}\left[\frac{1}{-2}\,1\,[5]\right] = \frac{1}{D}\left[\frac{-5}{2}\right] = \frac{-5x}{2}$$

$$y^{(5)} - y^{(3)} = 2x^2$$

$$y = \frac{1}{D^3}\,\frac{1}{D^2-1}[2x^2] = \frac{-1}{D^3}\left[\frac{1}{1-D^2}[2x^2]\right]$$

$$= \frac{-1}{D^3}\left[(1+D^2)[2x^2]\right]$$

$$= \frac{-1}{D^3}[2x^2+4]$$

$$= -\left(\frac{2x^5}{3\cdot4\cdot5} + \frac{4x^3}{1\cdot2\cdot3}\right)$$

$$= \frac{-x^5}{30} - \frac{2x^3}{3} \qquad \square$$

Digression: variation of parameters for 3rd order linear ODEs

Review: $y'' + f(x)y' + g(x)y = h(x)$

suppose y_1, y_2 are sol'ns of the homogeneous eq'n

then use ansatz $y = u_1 y_1 + u_2 y_2$

then $y' = \underbrace{u_1' y_1 + u_2' y_2}_{\text{we look for solution where this is 0}} + u_1 y_1' + u_2 y_2'$

$$u_1' y_1 + u_2' y_2 = 0$$

$$y'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

$$\Rightarrow y'' + f y' + g y = u_1' y_1' + u_2' y_2' + 0$$

$$u_1' y_1' + u_2' y_2' = h$$

For third order equations

$$y''' + f y'' + g y' + h y = k$$

suppose $y_1 \quad y_2 \quad y_3$ solve the homogeneous eqn

ansatz! $y = u_1 y_1 + u_2 y_2 + u_3 y_3$

$$y' = u_1 y_1' + u_2 y_2' + u_3 y_3' + \underbrace{u_1' y_1 + u_2' y_2 + u_3' y_3}_{=0}$$

$$u_1' y_1 + u_2' y_2 + u_3' y^3 = 0$$

$$y'' = u_1 y_1'' + u_2 y_2'' + u_3 y_3'' + \underbrace{u_1' y_1' + u_2' y_2' + u_3' y_3'}_{=0}$$

$$u_1' y_1' + u_2' y_2' + u_3' y_3' = 0$$

$$u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = h(x)$$

this is a 3x3 linear system for u_i' $\Rightarrow$ solve it integrate to get u_i

For fourth order systems get eqns:

$$u_1' y_1 + u_2' y_2 + u_3' y_3 + u_4' y_4 = 0$$

$$u_1' y_1' + u_2' y_2' + u_3' y_3' + u_4' y_4' = 0$$

$$u_1' y_1'' + u_2' y_2'' + u_3' y_3'' + u_4' y_4'' = 0$$

$$u_1' y_1''' + u_2' y_2''' + u_3' y_3''' + u_4' y_4''' = f$$

↑ the inhomogeneous part

Exponential shift theorem for inverse operators:

$$\frac{1}{P(D)}\left[u e^{ax}\right] = e^{ax} \frac{1}{P(D+a)}\left[u\right]$$

Proof: $e^{ax}u = e^{ax}\,P(D+a)\frac{1}{P(D+a)}[u]$

$$= P(D)\left[e^{ax}\frac{1}{P(D+a)}[u]\right]$$

$$e^{ax}u = P(D)\frac{1}{P(D)}[e^{ax}u]$$

$$\Rightarrow 0 = P(D)\left[e^{ax}\frac{1}{P(D+a)}[u] - \frac{1}{P(D)}[e^{ax}u]\right]$$

$$0 = \frac{1}{P(D)}0 = e^{ax}\frac{1}{P(D+a)}[u] - \frac{1}{P(D)}[e^{ax}u]$$

$$\Rightarrow \frac{1}{P(D)}[e^{ax}u] = e^{ax}\frac{1}{P(D+a)}[u] \quad \square$$

Example: $y'' - 2y' - 3y = x^2e^{2x}$

$$\frac{1}{D^2-2D-3}[x^2e^{2x}] = \frac{1}{(D-3)(D+1)}[x^2e^{2x}]$$

$$= e^{2x}\frac{1}{(D-1)(D+3)}[x^2]$$

$$= e^{2x}\frac{1}{D^2+2D-3}[x^2]$$

$$= \frac{-e^{2x}}{3}\,\frac{1}{1-\frac{2}{3}D-\frac{1}{3}D^2}[x^2] = (*)$$

$$\begin{array}{r|l} & 1+\frac{2}{3}x+\frac{7}{9}x^2+\dots \\ \hline 1-\frac{2}{3}x-\frac{1}{3}x^2 & 1 \\ & \underline{1-\frac{2}{3}x-\frac{1}{3}x^2} \\ & \frac{2}{3}x+\frac{1}{3}x^2 \\ & \underline{\frac{2}{3}x-\frac{4}{9}x^2+\dots} \\ & \frac{7}{9}x^2 \end{array}$$

$$\frac{3}{9}+\frac{4}{9}=\frac{7}{9}$$

$$(*) = \frac{-e^{2x}}{3}\left(1+\frac{2}{3}D+\frac{7}{9}D^2\right)[x^2]$$

$$= \frac{-e^{2x}}{3}\left(x^2+\frac{4}{3}x+\frac{14}{9}\right) \qquad \square$$

Example: $y''' - 5y'' + 8y' - 4y = 2e^{4x}x$

$$1 \qquad \frac{1}{\ }$$

$$\overline{D^3-5D^2+8D-4} = (D-1)(D-2)^2$$

$$\frac{1}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$= \frac{A(x^2-4x+4)+B(x^2-3x+3)+C(x-1)}{(x-1)(x-2)^2}$$

$$= \frac{(A+B)x^2+(-4A-3B+C)x+(4A+2B-C)}{(x-1)(x-2)^2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -4 & -3 & 1 & 0 \\ 4 & 2 & -1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \quad A=1 \quad B=-1 \quad C=1$$

$$\frac{1}{(D-1)(D-2)^3}\left[2e^{4x}x\right] = \left(\frac{1}{D-1} - \frac{1}{D-2} + \frac{1}{(D-2)^2}\right)\left[2e^{4x}x\right]$$

$$= e^{4x}\left(\frac{1}{D+3} - \frac{1}{D+2} + \frac{1}{(D+2)^2}\right)[2x]$$

$$= e^{4x}\left(\frac{1}{3}\,\frac{1}{1-(-D/3)} - \frac{1}{2}\,\frac{1}{1-(-D/2)} + \frac{1}{4}\,\frac{1}{1+D+1/4\,D^2}\right)[2x]$$

$$= e^{4x}\left(\frac{1}{3}\left(1-D/3\right) - \frac{1}{2}\left(1-D/2\right) + \frac{1}{4}\left(1-D\right)\right)[2x]$$

$$\frac{4-12+3}{12} = \frac{-5}{12}$$

$$= e^{4x}\left(x\left(\frac{2}{3} - 1 + \frac{1}{4}\right) + \left(-\frac{2}{9} + \frac{1}{2} - \frac{1}{2}\right)\right)$$

$$= e^{4x}\left(-\frac{2}{9} - \frac{5}{12}x\right) \quad \square$$

Laplace transforms

If $f(x)$ is defined for $x \geq 0$ then we define $L[f]$ to be the function

$$L[f](s) = \int_0^\infty e^{-sx} f(x)\,dx$$

Note L is linear in f, so this is an operator!

e.g.
$$L[1](s) = \int_0^\infty e^{-sx}\,dx$$

$$= \frac{-1}{s} e^{-sx} \Big|_0^\infty \quad (\text{for } s \geq 0)$$

$$= \frac{1}{s}$$