MAT 303 Lecture 8

Inhomogeneous systems:

$\vec{x}' = A\vec{x} + \vec{f}$ A need not be constant

suppose the homogeneous system $\vec{x}' = A\vec{x}$ has solutions $\vec{x}_1, \dots, \vec{x}_n$ (A is a n×n coefficient matrix)

ansatz: $\vec{x} = u_1\vec{x}_1 + \dots + u_n\vec{x}_n$ for $u_k(t)$ undetermined coefficient functions (variation of parameters)

$\vec{x}_k = \begin{bmatrix} x_k^{(1)} \\ \vdots \\ x_k^{(n)} \end{bmatrix}$ we are saying $\begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(n)} \end{bmatrix} = \begin{bmatrix} u_1 x_1^{(1)} + \dots + u_n x_n^{(1)} \\ \vdots \\ u_1 x_1^{(n)} + \dots + u_n x_n^{(n)} \end{bmatrix}$

$$= \underbrace{\begin{bmatrix} x_1^{(1)} & \cdots & x_n^{(1)} \\ \vdots & \ddots & \vdots \\ x_1^{(n)} & \cdots & x_n^{(n)} \end{bmatrix}}_{\text{fundemental matrix}} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = X\vec{u}$$

$\square = \square\,\square$

$X = \overset{\text{sol'n are columns}}{\begin{bmatrix} \vec{x}_1 & \dots & \vec{x}_n \end{bmatrix}}$

Since $\vec{x}_h' = A\vec{x}_k$, it folows that $\overset{\text{an equality of n×n matricies}}{X' = AX}$

ansatz: $\vec{x} = X\vec{u}$

$$\vec{x}' = X'\vec{u} + X\vec{u}'$$

$$A\vec{x} + \vec{f} = AX\vec{u} + \vec{f} = X'\vec{u} + \vec{f}$$

So the equation is solved when $X'\vec{u} + X\vec{u}' = X'\vec{u} + \vec{f}$

$\iff X\vec{u}' = \vec{f}$

Fact: if $\vec{x}_1, \dots, \vec{x}_n$ are linearly indep functions, then they will be pointwise lin indep.

Thus $\vec{u}' = X^{-1}\vec{f}$ $\Rightarrow$ reduces to n separate integrals

Example:

$$x' = x + 2y$$
$$y' = 4x - y + \frac{36e^{9t}}{e^{12t}+1} \quad \rightsquigarrow \quad A = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \quad \vec{f} = \begin{bmatrix} 0 \\ \frac{36e^{9t}}{e^{12t}+1} \end{bmatrix}$$

First for $A\vec{x} = \vec{x}'$, $\det(A - \lambda I) = (1-\lambda)(-1-\lambda) - 8$

$$= \lambda^2 - 1 - 8 = \lambda^2 - 9 = (\lambda-3)(\lambda+3)$$

$\lambda = 3$ $\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 4 & -4 & 0 \end{array}\right] \Rightarrow -2a + 2b = 0 \Rightarrow$ let $a=1$ then $b=1$

so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an ev.

$$\lambda = -3 \quad \left[\begin{array}{cc|c} 4 & 2 & 0 \\ 4 & 2 & 0 \end{array}\right] \Rightarrow \quad 4a + 2b = 0 \Rightarrow$$

let $a = 1$
then $b = -2$
so $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an e.v.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \xmapsto{A-3I} \vec{0} \qquad \begin{bmatrix} 1 \\ -2 \end{bmatrix} \xmapsto{A+3I} \vec{0}$$

$$X = \begin{bmatrix} e^{3t} & e^{-3t} \\ e^{3t} & -2e^{-3t} \end{bmatrix}$$

want $X \vec{u}' = \begin{bmatrix} 0 \\ \dfrac{36e^{9t}}{e^{12t}+1} \end{bmatrix}$

$$X^{-1} = \frac{1}{-2-1} \begin{bmatrix} -2e^{-3t} & -e^{-3t} \\ -e^{3t} & e^{3t} \end{bmatrix}$$

$$\vec{u}' = \frac{-1}{3} \begin{bmatrix} -2e^{-3t} & -e^{-3t} \\ -e^{3t} & e^{3t} \end{bmatrix} \begin{bmatrix} 0 \\ \dfrac{36e^{9t}}{e^{12t}+1} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{12e^{6t}}{e^{12t}+1} \\ \dfrac{-12e^{12t}}{e^{12t}+1} \end{bmatrix} \Rightarrow$$

$$u_1 = \int \frac{12e^{6t}}{e^{12t}+1}\,dt \quad \left[\begin{array}{l} \text{let } u = e^{6t} \\ du = 6e^{6t}\,dt \end{array}\right]$$

$$= 2\int \frac{du}{u^2+1} = 2\arctan(u) + C$$

$$= 2\arctan(e^{6t}) + C$$

$$u_2 = -\int \frac{12e^{12t}}{e^{12t}+1}\,dt \quad \begin{bmatrix} \text{Let } u = e^{12t}+1 \\ du = 12e^{12t}\,dt \end{bmatrix}$$

$$= -\int \frac{du}{u} = -\log(u) + D = -\log(e^{12t}+1) + D$$

$$\vec{x} = 2\arctan(e^{6t})e^{3t}\begin{bmatrix}1\\1\end{bmatrix} - \log(e^{12t}+1)e^{-3t}\begin{bmatrix}1\\-2\end{bmatrix}$$

$$+ Ce^{3t}\begin{bmatrix}1\\1\end{bmatrix} + De^{-3t}\begin{bmatrix}1\\-2\end{bmatrix}$$

$$\Rightarrow \left\{\begin{aligned} x &= 2\arctan(e^{6t})e^{3t} - \log(e^{12t}+1)e^{-3t} \\ &\qquad + Ce^{3t} + De^{-3t} \\ y &= 2\arctan(e^{6t})e^{3t} + 2\log(e^{12t}+1)e^{-3t} \\ &\qquad + Ce^{3t} - 2De^{-3t} \end{aligned}\right. \qquad \square$$

3x3 System:

$$x' = 2x - y - z + e^{t}$$

$$y' = x \qquad - z$$

$$z' = x - y \qquad + e^{-t}$$

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \qquad \vec{f} = \begin{bmatrix} e^{t} \\ 0 \\ e^{-t} \end{bmatrix}$$

cofactor

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 & -1 \\ 1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ -\lambda & 1 \end{vmatrix} - \begin{vmatrix} 1 & -\lambda \\ 1 & -1 \end{vmatrix}$$

$$= (2-\lambda)(\lambda^2 - 1) - (-1 + \lambda) - (-1 + \lambda)$$

$$= 2\lambda^2 - 2 - \lambda^3 + \lambda + 2 - 2\lambda$$

$$= -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda - 1)^2$$

$\lambda = 0$

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{aligned} 2a - b - c &= 0 \\ b - c &= 0 \\ 0 &= 0 \end{aligned}$$

Let $a = 1$ $b = 1$ $c = 1$ so $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an e.v.

$\lambda = 1$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right] \Rightarrow \quad a - b - c = 0$$

let $b = 1$ and $c = 0$ $\Rightarrow a = 1 \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

let $b = 0$ and $c = 1$ $\Rightarrow a = 1 \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$X = \begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & e^t & e^t & 1 & 0 & 0 \\ 1 & e^t & 0 & 0 & 1 & 0 \\ 1 & 0 & e^t & 0 & 0 & 1 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & e^t & e^t & 1 & 0 & 0 \\ 0 & 0 & -e^t & -1 & 1 & 0 \\ 0 & -e^t & 0 & -1 & 0 & 1 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & e^t & e^t & 1 & 0 & 0 \\ 0 & 1 & 0 & e^{-t} & 0 & -e^{-t} \\ 0 & 0 & 1 & e^{-t} & -e^{-t} & 0 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & e^{-t} & 0 & -e^{-t} \\ 0 & 0 & 1 & e^{-t} & -e^{-t} & 0 \end{array}\right]$$

$$X^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ e^{-t} & 0 & -e^{-t} \\ e^{-t} & -e^{-t} & 0 \end{bmatrix}$$

$$\vec{u}' = X^{-1}\vec{f} = \begin{bmatrix} -1 & 1 & 1 \\ e^{-t} & 0 & -e^{-t} \\ e^{-t} & -e^{-t} & 0 \end{bmatrix} \begin{bmatrix} e^{t} \\ 0 \\ e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} -e^{t} + e^{-t} \\ 1 - e^{-2t} \\ 1 \end{bmatrix} \Rightarrow$$

$$u_1 = \int (-e^{t} + e^{-t})\,dt$$

$$= -e^{t} - e^{-t} + C$$

$$u_2 = \int (1 - e^{-2t})\,dt$$

$$= t + \frac{1}{2}e^{-2t} + D$$

$$u_3 = \int dt = t + E$$

$$\vec{x} = X\vec{u}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix} \begin{bmatrix} -e^t - e^{-t} + C \\ t + \frac{1}{2}e^{-2t} + D \\ t + E \end{bmatrix}$$

$$-e^t - e^{-t} + e^t\left(t + \frac{1}{2}e^{-2t}\right) + e^t t$$

$$= e^t(2t-1) - \frac{1}{2}e^{-t}$$

$$-e^t - e^{-t} + e^t\left(t + \frac{1}{2}e^{-2t}\right)$$

$$= e^t(t-1) - \frac{1}{2}e^{-t}$$

$$-e^t - e^{-t} + te^t = e^t(t-1) - e^{-t}$$

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} e^t(2t-1) - \frac{1}{2}e^{-t} \\ e^t(t-1) - \frac{1}{2}e^{-t} \\ e^t(t-1) - e^{-t} \end{bmatrix} + \begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} \right.$$

$$\square$$

Matrix exponentials:

forget about this for now...

$$y' = \alpha y \Rightarrow y = C e^{\alpha t}$$

for constant coefficient linear systems

$$X' = AX \Rightarrow X = e^{At}$$ Huh? Does this make sense?

Well $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ ⇝ converges everywhere

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$$ ⇝ converges everywhere

$$(e^{At})' = \sum_{n=1}^{\infty} \frac{A^n t^{n-1}}{(n-1)!} = A\left(\sum_{n=1}^{\infty} \frac{A^{n-1} t^{n-1}}{(n-1)!}\right)$$

$$\overset{k=n-1}{=} A\left(\sum_{k=0}^{\infty} \frac{A^k t^k}{k!}\right) = Ae^{At}$$

Thus e^{At} satisfies $X' = AX$

Theorem: Let A, B be $m \times m$ matricies with $AB = BA$ (ie. A and B commute) then $e^{A+B} = e^A e^B$ $\left(\text{cor} = e^B e^A\right)$

Proof: Preliminary remark: $(A+B)^2 = (A+B)(A+B)$

$$= A^2 + AB + BA + B^2$$

$$\overset{\substack{\text{since}\\ [A,B]=0}}{=} A^2 + 2AB + B^2$$

Thus $(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$ (binomial theorem)

This only holds when $[A,B] = AB - BA = 0$

$$e^{A+B} = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{A^k B^{n-k}}{n!}$$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\boxed{n}} \frac{A^k}{k!} \frac{B^{(n-k)}}{(n-k)!} \overset{\substack{n=i+j\\ k=i}}{=} \sum_{i=0}^{\infty} \sum_{j=0}^{\boxed{\infty}} \frac{A^i}{i!} \frac{B^j}{j!}$$

$$= \left(\sum_{i=0}^{\infty} \frac{A^i}{i!} \right) \left(\sum_{j=0}^{\infty} \frac{B^j}{j!} \right) = e^A e^B \qquad \square$$

Cor: $e^{(t+s)A} = e^{tA} e^{sA}$ because $[tA, sA] = 0$

Thus $e^{tA} e^{-tA} = e^{0_{m\times m}} = \sum_{n=0}^{\infty} \frac{0^n}{n!} = \frac{0^0}{0!} = \frac{I}{1} = I$

But this means that e^{tA} is invertible

$\Rightarrow$ the columns are linearly independent solutions!

Example:

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda I + N \quad \text{where} \quad N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The cool thing about N is

$$N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad N^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

N is nilpotent, meaning $N^k = 0$ for some k (here $k=3$)

$$e^{At} = e^{\lambda t I + tN} \underset{\text{because } I \text{ commutes w. everything}}{=} e^{\lambda t I} e^{tN} = (*)$$

$$e^{\lambda t I} = \sum_{n=0}^{\infty} \frac{(\lambda t)^n I^n}{n!} = \left(\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \right) I = e^{\lambda t} I$$

$$e^{tN} = \sum_{n=0}^{\infty} \frac{N^n t^n}{n!} = I + Nt + N^2\frac{t^2}{2} = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$(*) = e^{\lambda t}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \qquad \square$$

In general if we have a m×m matrix of the form

→ this is a Jordan block

$$\underset{m\times m}{A =} \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} = \lambda I + N$$

where N has 1s on the super-diagonal

then $N^m = 0$ and N^k for $k < m$ has 1s on the kth super-diagonal (and 0s elsewhere)

then
$$e^{At} = e^{\lambda t I} e^{tN} = e^{\lambda t}\left(I + Nt + N^2\frac{t^2}{2} + \dots + N^{m-1}\frac{t^{m-1}}{(m-1)!}\right)$$

thus
$$e^{At} = \begin{bmatrix} 1 & t & t^2/2 & \dots & t^{m-1}/(m-1)! \\ & 1 & t & & \vdots \\ & & & \ddots & t \\ 0 & & & & 1 \end{bmatrix} e^{\lambda t}$$

In general to calculate matrix exponentials, it suffices to know how to exponentiate Jordan blocks

Observation: If $A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right]$ ↗ block decomposition

↳ square matrices

$$A^k = \left[\begin{array}{c|c} A_1^k & 0 \\ \hline 0 & A_2^k \end{array}\right]$$

So exponentials can be calculated block by block.

Last time:

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We had: (this is the structure of the matrix)

$$\underset{\vec v_4}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}} \overset{A+I}{\longmapsto} \underset{\vec v_3}{\begin{bmatrix}0\\2\\1\\0\end{bmatrix}} \overset{A+I}{\longmapsto} \underset{\vec v_2}{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}} \overset{A+I}{\longmapsto} \vec 0 \quad \Bigg| \quad \underset{\vec v_1}{\begin{bmatrix}1\\2\\1\\1\end{bmatrix}} \overset{A}{\longmapsto} \vec 0$$

$$A\vec v_1 = 0 \qquad A\vec v_2 = -\vec v_2$$

$$A\vec v_3 = \vec v_2 - \vec v_3 \qquad A\vec v_4 = \vec v_3 - \vec v_4$$

With respect to the basis $\vec v_1 \dots \vec v_4$, A becomes

$$J = \begin{array}{c|c|ccc|}
 & \vec v_1 & \vec v_2 & \vec v_3 & \vec v_4 \\
\vec v_1 & 0 & 0 & 0 & 0 \\ \cline{2-5}
\vec v_2 & 0 & -1 & 1 & 0 \\
\vec v_3 & 0 & 0 & -1 & 1 \\
\vec v_4 & 0 & 0 & 0 & -1
\end{array}$$

This is a Jordan decomposition

$$Q\vec e_i = \vec v_i$$

$$Q = \begin{bmatrix}\vec v_1 \dots \vec v_4\end{bmatrix} = \begin{bmatrix}1 & 1 & 0 & 0\\ 2 & 0 & 2 & 1\\ 1 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}$$

so the result is $A = \underbrace{Q^{-1}JQ}_{\text{conjugating by } Q}$

$$(Q^{-1}JQ)^2$$

$$= (Q^{-1}JQ)(Q^{-1}JQ) = Q^{-1}J^2Q$$

$$(Q^{-1}JQ)^n = Q^{-1}J^nQ$$

thus $e^{At} = e^{Q^{-1}JQt} = Q^{-1}e^{Jt}Q$

(do it for every term in the series)

So in the end, calculating the exponential in general is a tiny bit more work than our previous approach

We won't be solving problems this way

This is super useful numerical algorithm

By calculating the $\underbrace{\text{series to } \overset{\text{easy to implement}}{\text{some large}}\ N}$,

exact → $\underbrace{\text{(or exactly by } \overset{\text{hard to implement}}{\text{doing Jordan}}\text{)}}$] this gives an approximation that's easy to calculate

☆ And this basically finishes all of the material that is regularly part of this course! ☆

(Still need to do applications)