MAT 303 Lecture 7

HW 1 due at 11:59pm
today; submit via BB

Complex roots

$$A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \qquad \det(A - \lambda I) = (4-\lambda)^2 + 9$$

$$= \lambda^2 - 8\lambda + 16 + 9$$

$$= \lambda^2 - 8\lambda + 25$$

$$\lambda = 4 + 3i \qquad \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\lambda = \frac{8 \pm \sqrt{64 - 100}}{2}$$

$$= 4 \pm 3i$$

$$\left[\begin{array}{cc|c} -3i & -3 & 0 \\ 3 & -3i & 0 \end{array}\right]$$

multiples of one another

$$-3ia - 3b = 0$$

Let $a = 1$, $b = -i$ so $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector

$$\begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(4+3i)t} = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{4t}\left(\cos(3t) + i\sin(3t)\right) = (*)$$

Original equation is $A\vec{x} = \vec{x}'$ and A is real

so this equation implies $A(\mathrm{Re}\,\vec{x}) = (\mathrm{Re}\,\vec{x})'$

$$A(\mathrm{Im}\,\vec{x}) = (\mathrm{Im}\,\vec{x})'$$

$$(*) = e^{4t}\begin{bmatrix}\cos(3t)\\ \sin(3t)\end{bmatrix} + i\, e^{4t}\begin{bmatrix}\sin(3t)\\ -\cos(3t)\end{bmatrix}$$

Thus $e^{4t}\begin{bmatrix}\cos(3t)\\ \sin(3t)\end{bmatrix}$ and $e^{4t}\begin{bmatrix}\sin(3t)\\ -\cos(3t)\end{bmatrix}$

are solutions. $\Rightarrow \begin{cases} x = Ce^{4t}\cos(3t) + De^{4t}\sin(3t) \\ y = Ce^{4t}\sin(3t) - De^{4t}\cos(3t) \end{cases}$ $\boxtimes$

With the other eigenvector, suppose $\lambda_1 = \lambda_2$

$(A-\lambda_1 I)\vec{v} = \vec{0} \Rightarrow (A - \bar{\lambda}_1)\bar{\vec{v}} = \vec{0}$

Thus $\bar{\vec{v}}$ is an eigenvector for λ_2.

We said, let $\vec{v} = \underbrace{\vec{v}_1}_{\text{real part}} + i\,\underbrace{\vec{v}_2}_{\text{imaginary part}}$

Well we have a solution $\lambda_1 = \alpha + i\beta$

$$(\vec{v}_1 + i\vec{v}_2)e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$

$$= e^{\alpha t}\Big((\vec{v}_1\cos(\beta t) - \vec{v}_2\sin(\beta t)) + i(\vec{v}_1\sin(\beta t) + \vec{v}_2\cos(\beta t))\Big)$$

For $(\vec{v}_1 - i\vec{v}_2)\, e^{\alpha t} \left(\cos(\beta t) - i \sin(\beta t)\right)$

$$= e^{\alpha t}\Big(\big(\vec{v}_1 \cos(\beta t) - \vec{v}_2 \sin(\beta t)\big) - i\big(\vec{v}_1 \sin(\beta t) + \vec{v}_2 \cos(\beta t)\big)\Big)$$

Multiple roots:

Case I: $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad \det(A - \lambda I) = (2-\lambda)^2$

so $\lambda = 2$ is an eigenvalue of mult 2

$A - 2I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so in this case the kernel is 2D

so we take any two linearly indep eigenvectors

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ $\vec{v}_1 e^{2t}$ and $\vec{v}_2 e^{2t}$ are solutions

$$\Rightarrow \begin{cases} x = Ce^{2t} \\ y = De^{2t} \end{cases}$$ is the general solution ☒

Case II: $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $\det(A - \lambda I)$
$= (2-\lambda)^2$

so $\lambda = 2$ is still an eigenvalue of mult. 2

but now $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} b = 0 \\ 0 = 0 \end{array} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector

But there is no second l.i. eigenvector!

What do we do?

The original system was:

$$x' = 2x + y$$
$$y' = 2y \Rightarrow y = Ce^{2t}$$

$\frac{dy}{y} = 2d$
$\Rightarrow \log(y) = 2t + \log(C)$
$\Rightarrow y = Ce^{2t}$

$$\Rightarrow x' = 2x + Ce^{2t}$$

this is a first order linear eqn

$$\Rightarrow x' - 2x = Ce^{2t}$$

First we solve $x' = 2x \Rightarrow x = e^{2t}$

ansatz: $x = ue^{2t}$

$x' - 2x = Ce^{2t}$ becomes

$$u'e^{2t} + 2ue^{2t} - 2ue^{2t} = Ce^{2t}$$

$$\Rightarrow u' = C$$

$$\Rightarrow u = Ct + D$$

$$\Rightarrow \begin{cases} x = Cte^{2t} + De^{2t} \\ y = Ce^{2t} \end{cases}$$ is the general sol'n

Back to the eigenspace approach!

crux of the issue if $(\lambda - a)^m$ is a factor of $\det(A - \lambda I)$, then

$$\dim\left(\underbrace{\ker(A - \lambda I)^m}_{\text{generalised eigenspace}}\right) = m$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(A-2I)^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \qquad A-2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A-2I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A-2I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

ansatz: $\vec{x} = (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t}$

$$A\vec{x} = \vec{x}'$$

$$\vec{x}' = (\vec{v}_1 \lambda t + \vec{v}_1 + \vec{v}_2 \lambda)\, e^{\lambda t}$$

$$A\vec{x} = (t A\vec{v}_1 + A\vec{v}_2)\, e^{\lambda t}$$

$$\Rightarrow \quad A\vec{v}_1 = \lambda \vec{v}_1 \qquad \Rightarrow \quad (A-\lambda I)\vec{v}_1 = 0$$

$$A\vec{v}_2 = \vec{v}_1 + \lambda \vec{v}_2 \qquad\quad (A-\lambda I)\vec{v}_2 = \vec{v}_1$$

This means $\vec{v}_2 \xmapsto{A-\lambda I} \vec{v}_1 \xmapsto{A-\lambda I} \vec{0}$

So then $\vec{x} = te^{\lambda t}\vec{v}_1 + e^{\lambda t}\vec{v}_2$

is a solution!

$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ we had $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{matrix} b=0 \\ 0=0 \end{matrix}$

so let $a=1$

and $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an e.v.

now consider

$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{matrix} b=1 \\ 0=0 \end{matrix}$ let $a=0$, get $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{v}_2$

then $\vec{x}_1 = te^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ D

is a solution

also $\vec{x}_2 = e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ C thus the general sol'n is:

$$\begin{cases} x = Ce^{2t} + Dte^{2t} \\ y = De^{2t} \end{cases}$$

This matches our previous solution!

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \qquad \vec{V}_3 \xmapsto{A-\lambda I} V_2 \xmapsto{A-\lambda I} V_1 \xmapsto{A-\lambda I} \vec{0}$$

ansatz: $\vec{x} = \left(\frac{t^2}{2}\vec{v}_1 + t\vec{v}_2 + t\vec{v}_3\right)e^{\lambda t}$

$\vec{x}\,' = A\vec{x}$ becomes

$$\vec{x}\,' = \left(\frac{t^2}{2}\lambda\vec{v}_1 + t\left(\vec{v}_1 + \lambda\vec{v}_2\right) + \left(\vec{v}_2 + \lambda\vec{v}_3\right)\right)e^{\lambda t}$$

$$A\vec{x} = \left(\frac{t^2}{2}A\vec{v}_1 + tA\vec{v}_2 + A\vec{v}_3\right)e^{\lambda t}$$

$$\Rightarrow \quad A\vec{v}_1 = \lambda \vec{v}_1 \qquad (A-\lambda I)\vec{v}_1 = \vec{0}$$

$$A\vec{v}_2 = \lambda \vec{v}_2 + \vec{v}_1 \quad \Rightarrow \quad (A-\lambda I)\vec{v}_2 = \vec{v}_1$$

$$A\vec{v}_3 = \lambda \vec{v}_3 + \vec{v}_2 \qquad (A-\lambda I)\vec{v}_3 = \vec{v}_2$$

In general if you can find

$$\vec{v}_k \overset{A-\lambda I}{\longmapsto} \vec{v}_{k-1} \overset{A-\lambda I}{\longmapsto} \dots \overset{A-\lambda I}{\longmapsto} v_1 \overset{A-\lambda I}{\longmapsto} 0$$

$$\vec{x}_1 = \vec{v}_1 e^{\lambda t}$$

$$\vec{x}_2 = (t\vec{v}_1 + \vec{v}_2)e^{\lambda t}$$

$$\vdots$$

$$\vec{x}_k = \left(\frac{t^{k-1}}{(k-1)!}\vec{v}_1 + \frac{t^{k-2}}{(k-2)!} + \dots + \vec{v}_{k-1}t + \vec{v}_k\right)e^{\lambda t}$$

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \det(A-\lambda I) \quad \text{cofactor expansion}$$

$$(2-\lambda)\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} + 0 + 0 + 0$$

$$= (*)$$

Cofactor expansion

$$\begin{vmatrix} a_{1,1} & \cdots & a_{1,m} \\ a_{2,1} & & \vdots \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{vmatrix} = a_{1,1} \begin{vmatrix} a_{2,2} & \cdots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{m,m} & \cdots & a_{m,m} \end{vmatrix} \Rightarrow (n-1)\times(m-1)$$

$$+\, a_{2,1} \begin{vmatrix} a_{3,2} & \cdots & a_{3,m} \\ \vdots & \ddots & \vdots \\ a_{m,2} & \cdots & a_{m,n} \\ a_{1,2} & \cdots & a_{1,m} \end{vmatrix}$$

(can be done on any column or row)

$$+ \dots$$

$$+\, a_{m,1} \begin{vmatrix} a_{1,2} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{m-1,2} & \cdots & a_{n-1,m} \end{vmatrix}$$

$$(*) = (2-\lambda) \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^4$$

So 2 is an eigenvalue of mult. 4

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{c} b=0 \\ c=0 \\ d=0 \\ 0=0 \end{array} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is an eigenvector

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} b=1 \\ c=0 \\ d=0 \\ 0=0 \end{array} \Rightarrow \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} b=0 \\ c=1 \\ d=0 \\ 0=0 \end{array} \Rightarrow \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} b=0 \\ c=0 \\ d=1 \\ 0=0 \end{array} \Rightarrow \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so some solutions are:

$$\vec{x}_1 = \vec{v}_1 e^{2t}$$

C

$$\vec{x}_2 = \left(t\vec{v}_1 + \vec{v}_2\right) e^{2t}$$

D

$$\vec{x}_3 = \left(\frac{t^2}{2}\vec{v}_1 + t\vec{v}_2 + \vec{v}_3\right)e^{2t} \qquad E$$

$$\vec{x}_4 = \left(\frac{t^3}{6}\vec{v}_1 + \frac{t^2}{2}\vec{v}_2 + t\vec{v}_3 + \vec{v}_4\right)e^{2t} \qquad F$$

$$\begin{cases} x = \left(C + Dt + E\frac{t^2}{2} + F\frac{t^3}{6}\right)e^{2t} \\ y = \left(D + Et + F\frac{t^2}{2}\right)e^{2t} \\ z = (E + Ft)e^{2t} \\ w = Fe^{2t} \end{cases}$$

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \det(A - \lambda I)$$

$$= (-1-\lambda)\begin{vmatrix} 1-\lambda & -4 & 2 \\ 1 & -3-\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} + 0$$

$$= -(\lambda+1)\left(-\lambda\begin{vmatrix} 1-\lambda & -4 \\ 1 & -3-\lambda \end{vmatrix} + 0\right)$$

$$= -(\lambda+1)(-\lambda)\left((1-\lambda)(-3-\lambda)+4\right)$$

$$= (\lambda+1)\lambda\left(\lambda^2+2\lambda+1\right)$$

$$= (\lambda+1)^3\lambda$$

with λ mult 3

I) $\vec{v}_1 \mapsto 0$
$\vec{v}_2 \mapsto 0$
$\vec{v}_3 \mapsto 0$

II) $\vec{v}_1 \mapsto 0$
$\vec{v}_3 \mapsto \vec{v}_2 \mapsto 0$

III) $\vec{v}_3 \mapsto \vec{v}_2 \mapsto \vec{v}_1 \mapsto \vec{0}$

$\lambda = 0$

$$\left[\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{aligned} a - d &= 0 \\ b - 2d &= 0 \\ c - d &= 0 \\ 0 &= 0 \end{aligned}$$

setting $d=1$ $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$

eigenvector for $\lambda = 0$

$\lambda = -1$

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -4 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \begin{aligned} c &= 0 \\ 0 &= 0 \\ b - 2c + d &= 0 \\ d &= 0 \end{aligned}$$

so $b = c = d = 0$

let $a = 1$, then $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \vec{v}_1$ is an e.v.

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & -4 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{c} c = 1 \\ 0 = 0 \\ b - 2c + d = 0 \\ d = 0 \end{array} \Rightarrow \begin{array}{c} b = 2 \\ c = 1 \\ d = 0 \end{array} \quad \text{let } a = 0 \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -4 & 2 & 2 \\ 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{c} c = 0 \\ 0 = 0 \\ b - 2c + d = 1 \end{array} \Rightarrow \begin{array}{c} b = 1 \\ c = 0 \\ d = 0 \end{array} \quad \text{let } a = 0 \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$d = 0$$

$$\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \xmapsto{A+I} \begin{bmatrix}0\\2\\1\\0\end{bmatrix} \xmapsto{A+I} \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \xmapsto{A+I} \vec{0} \quad \Bigg| \quad \begin{bmatrix}1\\2\\1\\1\end{bmatrix} \xmapsto{A} \vec{0}$$

C $$\vec{x}_1 = \begin{bmatrix}1\\2\\1\\1\end{bmatrix}$$

D $$\vec{x}_2 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} e^{-t}$$

E $$\vec{x}_3 = te^{-t}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + e^{-t}\begin{bmatrix}0\\2\\1\\0\end{bmatrix}$$

F $$\vec{x}_4 = \frac{t^2}{2}e^{-t}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + te^{-t}\begin{bmatrix}0\\2\\1\\0\end{bmatrix} + e^{-t}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$$\begin{cases} x = C + e^{-t}\left(D + Et + F\,t^2/2\right) \\ y = 2C + e^{-t}\left(2E + F(2t+1)\right) \\ z = C + e^{-t}\left(E + Ft\right) \end{cases}$$

the general solution

$w = C$

Inhomogeneous linear systems next time

Operators

An operator $\mathcal{L}$ is something that takes functions to functions and is linear, that is

$$\mathcal{L}[f + \lambda g] = \mathcal{L}[f] + \lambda \mathcal{L}[g]$$

e.g. $\mathcal{L} = D = \frac{d}{dx}$ differentiation is linear

If $P(x) = a_n x^n + \dots + a_0$, then

$$P(D) = a_n D^n + \dots + a_0$$ is an operator

called polynomial (differential) operator

e.g. the equation $y'' + 2y' + y = f(x)$

is $P(D)[y] = f(x)$

for $P(x) = x^2+2x+1$

Theorem: If P and Q are polynomials, then

$$(P\cdot Q)(D) = P(D)\circ Q(D) \qquad \square$$

example: $x^2+2x+1 = (x+1)(x+1)$

so $D^2+2D+1 = (D+1)(D+1)$

Cor: $P(D)\,Q(D) = (P\cdot Q)(D) = (Q\cdot P)(D)$

$= Q(D)\,P(D)$

example: for $(x+1)(x+2)$

$$(D+1)(D+2) = (D+2)(D+1)$$

works with D replaced by $\mathcal{L}$, so long as there is no f

example: Compute $(D^2-2D-3)\left[\sin(x)+x^2\right]$

Method 1: $(D^2-2D-3)[\sin(x)]$

$+ (D^2-2D-3)[x^2]$

$= -\sin(x) - 2\cos(x) - 3\sin(x)$

$$+ 2 - 4x - 3x^2$$

$$= -3x^2 - 4x + 2 - 2\cos(x) - 4\sin(x)$$

Method 2: $(D^2 - 2D - 3)\left[\sin(x) + x^2\right]$

$$= (D-3)(D+1)\left[\sin(x) + x^2\right]$$

$$= (D-3)\left[\cos(x) + 2x + \sin(x) + x^2\right]$$

$$= -\sin(x) + 2 + \cos(x) + 2x$$

$$-3\left(\cos(x) + 2x + \sin(x) + x^2\right)$$

$$= -3x^2 - 4x + 2 - 2\cos(x) - 4\sin(x) \boxtimes$$

Theorem (Exponential shift):

$$P(D)\left[u e^{ax}\right] = e^{ax} \cdot P(D+a)\left[u\right]$$

Proof: (next time) $\square$

Example: $(D^2 + 2D + 3)\left[e^{2x}\sin(x)\right]$

$$= e^{2x}\left((D+2)^2 + 2(D+2) + 3\right)\left[\sin(x)\right]$$

$$= e^{2x}\left(D^2+6D+11\right)[\sin(x)]$$

$$= e^{2x}\left(-\sin(x)+6\cos(x)+11\sin(x)\right)$$

$$= e^{2x}\left(6\cos(x)+10\sin(x)\right) \quad \square$$