MAT 303 Lecture 5

Recall: For $y' + f(x)y = g(x)$

Suppose y_1 is a solution of $y' + f(x)y = 0$,

then we used the ansatz $y = uy_1$ for u an

unknown function.

For second order systems:

$$y'' + f(x)y' + g(x)y = h(x)$$

If y_1 and y_2 are solutions of $y'' + fy' + gy = 0$

use the ansatz $y = u_1 y_1 + u_2 y_2$

for two unknown functions u_1, u_2

Then $y' = \boxed{u_1' y_1 + u_2' y_2} + u_1 y_1' + u_2 y_2'$

Because we have an extra degree of freedom

we will search for u_1, u_2, s.t. $\boxed{u_1' y_1 + u_2' y_2 = 0}$

$$y'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$$

Thus: $h = y'' + f y' + g y = (u_1 y_1'' + u_1 f y_1' + u_1 g y_1)$

$$+ (u_2 y_2'' + u_2 f y_2' + u_2 g y_2)$$

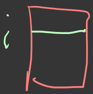
$$+ u_1' y_1' + u_2' y_2'$$

$$= u_1 (\overbrace{y_1'' + f y_1' + g y_1}^{0}) + u_2 (\overbrace{y_2'' + f y_2' + g y_2}^{0})$$

$$+ u' y_1' + u_2' y_2'$$

In summary:

$$\boxed{\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= h \end{aligned}}$$



$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ h \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

exists iff

$ad-bc = 0$

in our example $W = y_1 y_2' - y_2 y_1'$

if y_1, y_2 lin indep solutions, then $W \neq 0$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ h \end{bmatrix}$$

$$= \frac{1}{y_1 y_2' - y_1' y_2} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ h \end{bmatrix}$$

$$= \frac{1}{y_1 y_2' - y_1' y_2} \begin{bmatrix} -y_2 h \\ y_1 h \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} u_1' &= \frac{-1}{w} y_2 h \\ u_2' &= \frac{1}{w} y_1 h \end{aligned}}$$

$$\Rightarrow \begin{aligned} u_1 &= -\frac{1}{w}\int y_2(x)\, h(x)\, dx + C \\ u_2 &= \frac{1}{w}\int y_1(x)\, h(x)\, dx + D \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} y &= u_1 y_1 + u_2 y_2 = \\ &-y_1 \frac{1}{W}\int y_2(x)h(x)\,dx + y_2 \frac{1}{W}\int y_1(x)h(x)\,dx \\ &+ C y_1 + D y_2 \end{aligned}}$$

Examples:

$$y'' + y = \frac{1}{\cos^3(x)}$$

For $y'' + y = 0$ consider $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

so get $y_1 = \cos(x)$ $y_2 = \sin(x)$

Then use the ansatz $y = u_1 \cos(x) + u_2 \sin(x)$

such that $u_1' \cos(x) + u_2' \sin(x) = 0$

then note that $y_1 y_2' - y_1' y_2 = \underbrace{\cos^2(x) + \sin^2(x)}_{=1}$

so we have

$$u_1' = \frac{-1}{1} \sin(x) \frac{1}{\cos^3(x)} = -\tan(x)\sec^2(x)$$

$$u_2' = \frac{1}{1}\cos(x)\frac{1}{\cos^3(x)} = \sec^2(x)$$

$$u_1 = -\int \tan(x)\sec^2(x)\,dx \quad \left[\begin{array}{l} \text{Let } u = \tan(x) \\ du = \sec^2(x)\,dx \end{array}\right]$$

$$= -\int u\,du = \frac{-1}{2}u^2 + C = \frac{-1}{2}\tan^2(x) + C$$

$$u_2 = \int \sec^2(x)\,dx = \tan(x) + D$$

Thus $y = \frac{-1}{2}\frac{\sin^2(x)}{\cos(x)} + \frac{\sin^2(x)}{\cos(x)} + C\cos(x) + D\sin(x)$

$$= \frac{1}{2}\tan(x)\sin(x) + C\cos(x) + D\sin(x) \quad \square$$

$$y'' - y = \frac{2}{e^x + 1}$$

Consider $y'' - y = 0$, so $\lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$

and we get $y_1 = e^x$ $\quad y_2 = e^{-x}$

Use the ansatz $y = u_1 e^x + u_2 e^{-x}$

with $u_1' e^x + u_2' e^{-x} = 0$, then

for $W = y_1 y_2' - y_1' y_2 = -e^{x} e^{-x} - e^{x} e^{-x} = -2$

Then $u_1' = \frac{-1}{-2} e^{-x} \frac{2}{e^{x}+1} = \frac{e^{-x}}{e^{x}+1} = \frac{(e^{x}+1)e^{-x}-1}{e^{x}+1}$

$= e^{-x} + \frac{-e^{-x}}{1+e^{-x}}$

$$u_2' = \frac{1}{-2} e^{x} \frac{2}{e^{x}+1} = \frac{-e^{x}}{e^{x}+1}$$

$$u_1 = \int \left[e^{-x} + \frac{-e^{-x}}{1+e^{-x}} \right] dx = -e^{-x} + \log(1+e^{-x}) + C$$

$$u_2 = -\int \frac{e^{x}}{e^{x}+1} dx = -\log(1+e^{x}) + D$$

$y = u_1 y_1 + u_2 y_2 =$

$-1 + e^{x} \log(1+e^{-x}) - e^{-x} \log(1+e^{x}) + Ce^{x} + De^{-x}$

$$x^2 y'' + x y' - y = 4x^2 e^{x}$$

The equation $x^2 y'' + x y' - y = 0$ is an Euler equation, so we consider

$$\lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

and get particular solutions $y_1 = x$ $y_2 = x^{-1}$

So use the ansatz that

$$y = u_1 x + u_2 x^{-1}$$

with $u_1' x + u_2' x^{-1} = 0$, than we have

$$W = y_1 y_2' - y_1' y_2 = \frac{-x}{x^2} - \frac{1}{x} = \frac{-2}{x}$$

for $x^2 y'' + x y' - y = 4x^2 e^x$

$$\Rightarrow y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 4e^x \quad , \quad h = 4e^x$$

and $u_1' = \frac{-1}{-2/x} 1/x \, 4e^x = 2e^x$

$$u_2' = \frac{1}{-2/x} x \, 4e^x = -2x^2 e^x$$

$$u_1 = 2e^x + C$$

$$u_2 = -2\int x^2 e^x\,dx \quad \left[\begin{array}{ll} f = x^2 & dg = e^x dx \\ df = 2x\,dx & g = e^x \end{array}\right]$$

$$= -2\left(x^2 e^x - 2\int x e^x\,dx\right) \quad \left[\begin{array}{ll} f = x & dg = e^x dx \\ df = dx & g = e^x \end{array}\right]$$

$$= -2\left(x^2 e^x - 2\left(xe^x - \int e^x\,dx\right)\right)$$

$$= -2x^2 e^x + 4xe^x - 4e^x + D$$

$$y = u_1 y_1 + u_2 y_2 = 2xe^x - 2xe^x + 4e^x - 4x^{-1}e^x + Cx + Dx^{-1}$$

$$= 4e^x - 4x^{-1}e^x + Cx + Dx^{-1} \qquad \blacksquare$$

Undetermined coefficients

constant coef. linear eqn's with

$$ay'' + by' + cy = h \qquad \text{for } h = de^{tx}$$

Case 1: $h = de^{tx}$ for t not a root of $a\lambda^2 + b\lambda + c = 0$

Ansatz $y = Ae^{tx}$

$$ay'' + by' + cy = A(at^2 + bt + c)e^{tx} = de^{tx}$$

thus $A = \dfrac{d}{at^2 + bt + c}$

Case 2 & 3:

If t is a simple root, use $y = Axe^{tx}$

If t is a double root, use $y = Ax^2e^{tx}$

If $a\lambda^2 + b\lambda + c = a(\lambda - t)(\lambda - s)$ $\quad t \neq s$

then $ay'' + by' + cy = a(D-s)(D-t)y$ $\quad$ where $D = \frac{d}{dx}$

$$(D-t)(xe^{tx}) = e^{tx} + txe^{tx} - txe^{tx} = e^{tx}$$

Then $(D-s)e^{tx} = te^{tx} - se^{tx} = (t-s)e^{tx}$

$$(D-t)^2(x^2e^{tx}) = (D-t)(2xe^{tx} + tx^2e^{tx} - tx^2e^{tx})$$

$$= (D-t)(2xe^{tx}) = 2e^{tx}$$

Examples:

$$y'' - 4y' + 3y = 24e^{-x}$$

For $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$

Ansatz $y = Ae^{-x}$

$$y'' - 4y' + 3y = A\left((-1)^2 - 4(-1) + 3\right)e^{-x} = 24e^{-x}$$

$$\Rightarrow 8A = 24 \Rightarrow A = 3$$

Thus the general solution is: $y = 3e^{-x} + Ce^{3x} + De^{x}$ ∎

$$y'' - 4y' + 3y = 8\cos(x)$$

$$\mathrm{Re}(\alpha + i\beta) = \alpha$$

$$\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} = \mathrm{Re}\left(e^{ix}\right)$$

We will solve for $y'' - 4y' + 3y = 8e^{ix}$

and then $8\cos(x) = \mathrm{Re}(8e^{ix})$

$$= \mathrm{Re}(y'' - 4y' + 3y) = (\mathrm{Re}(y))'' - 4(\mathrm{Re}(y))' + 3\,\mathrm{Re}(y)$$

So $\mathrm{Re}(y)$ is a solution to our eq'n,

Ansatz $y = Ae^{ix}$

Then $y'' - 4y' + 3y = A(i^2 - 4i + 3)e^{ix} = 8e^{ix}$

$$\Rightarrow A = \frac{8}{2-4i} = \frac{4}{1-2i} = \frac{4(1+2i)}{(1-2i)(1+2i)} = \frac{4+8i}{5}$$

$$y = \frac{4+8i}{5}e^{ix} = \frac{4+8i}{5}(\cos(x) + i\sin(x))$$

$$\mathrm{Re}(y) = \frac{4}{5}\cos(x) - \frac{8}{5}\sin(x)$$

The general solution is

$$y = \frac{4}{5}\cos(x) - \frac{8}{5}\sin(x) + Ce^{3x} + De^{x}$$

Systems of ODEs:

A first order ODE system is of the form:

$$\begin{aligned} y_1' &= f_1(x, y_1, \dots, y_m) \\ y_2' &= f_2(x, y_1, \dots, y_m) \\ &\vdots \\ y_m' &= f_m(x, y_1, \dots, y_m) \end{aligned}$$

Reduction: Suppose we have a n-th order ODE

$$y^{(n)} = f(x, y, \dots, y^{(n-1)})$$

Consider

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \end{aligned}$$

$$y_n = f(x, y_1, y_2, \ldots, y_n)$$

Linear first order systems:

$$y_1' = a_{11}(x)\, y_1 + \ldots + a_{1n}(x)\, y_n + f_1(x)$$

$$\vdots$$

$$y_n' = a_{n1}(x)\, y_1 + \ldots + a_{nn}(x)\, y_n + f_n(x)$$

$$A(x) = \begin{bmatrix} a_{11}(x) & \cdots & a_{1n}(x) \\ \vdots & \ddots & \vdots \\ a_{n1}(x) & \cdots & a_{nn}(x) \end{bmatrix} \qquad \vec{f}(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

then for $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{bmatrix}$

the equation becomes

$\vec{y}\,' = A\vec{y} + \vec{f}$ an equation for the unknown vector-valued function $\vec{y}$

More linear algebra:

a m×n matrix

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ i \quad x \; x & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \end{array} \quad \begin{array}{cc} & k \\ n & \begin{bmatrix} b_{11} & \cdots & x & \cdots & b_{1k} \\ \vdots & & x & & \vdots \\ b_{n1} & \cdots & j & \cdots & b_{nk} \end{bmatrix} \end{array}$$

$$= \begin{array}{cc} & k \\ m & \begin{bmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mk} \end{bmatrix} \end{array}$$

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}$$

$$\begin{array}{cc} & 3 \\ 2 & \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{cc} & 1 \\ 3 & \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \end{array} = \begin{array}{cc} & 1 \\ 2 & \begin{bmatrix} 2\times1 + 3\times2 + 1\times1 \\ 4\times1 + 1\times2 + 0\times1 \end{bmatrix} \end{array} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

$$A^{-1}A = AA^{-1} = I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$