MAT 303 Lecture 4

Second Order Linear ODEs continued:

General result: For $y'' + f(x)y' + g(x)y = 0$

if f, g are continuous (locally lipschitz) on an interval I

then the space of solutions to this eq'n is a 2-D vector space

i.e. there are solutions y_1, y_2 st neither is a multiple of the other and the general solution is

$y = Cy_1 + Dy_2$.

In particular: suppose the equation is $ay'' + by' + cy = 0$ for $a, b, c \in \mathbb{R}$, $a \neq 0$.

Ansatz: try $y = e^{tx}$, eq'n becomes

$$at^2e^{tx} + bte^{tx} + ce^{tx} = 0 \Rightarrow at^2 + bt + c = 0$$

Case 1: $\to$ $b^2 - 4ac > 0$ Suppose $a\lambda^2 + b\lambda + c = 0$ has two distinct real roots t_1, t_2. Then e^{t_1x} and e^{t_2x} are sol'ns

By the theorem, all solutions have the form $Ce^{t_1 x} + De^{t_2 x}$.

$b^2 - 4ac < 0$

Case 2: $\Rightarrow$ Suppose $a\lambda^2 + b\lambda + c = 0$ has a pair of conjugate roots $\alpha \pm i\beta$ for $\beta \neq 0$.

Well $e^{(\alpha + i\beta)x}$ and $e^{(\alpha - i\beta)x}$ are solutions, but these aren't real.

$$e^{(\alpha + i\beta)x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$

$$\Rightarrow Ce^{(\alpha+i\beta)x} + De^{(\alpha - i\beta)x}$$

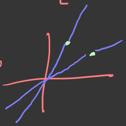
$$= e^{\alpha x}\left(C(\cos(\beta x) + i\sin(\beta x)) + D(\cos(-\beta x) + i\sin(-\beta x))\right)$$

$$= e^{\alpha x}\left((C+D)\cos(\beta x) + i(C-D)\sin(\beta x)\right) = (*)$$

If we let $A = C + D$, $B = i(C - D)$, then

$$(*) = Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x)$$

Takeaway: $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ are linearly independent solutions, thus by the thm the general sol'n is



$$Ce^{\alpha x}\cos(\beta x) + De^{\alpha x}\sin(\beta x)$$

$\Rightarrow b^2-4ac=0$

Case 3: $a\lambda^2+b\lambda+c=0$ has a multiplicity two real root

Well if this root is t, then e^{tx} is a solution.

When do we find the second solution?

Suppose the equation was $(\lambda-t)(\lambda-t^*)=0$ for $t^*\neq t$. Then e^{tx} and e^{t^*x} are solutions.

Thus, so is $\dfrac{e^{t^*x}-e^{tx}}{t^*-t}$. Now take $\lim_{t^*\to t}$

$$\lim_{t^*\to t}\frac{e^{t^*x}-e^{tx}}{t^*-t}\overset{\text{L'Hop}}{=}\lim_{t^*\to t}\frac{xe^{t^*x}}{1}=xe^{tx}.$$

Another way: Consider $(\lambda-(t+i\beta))(\lambda-(t-i\beta))$ $\beta\neq0$

Then $e^{tx}\sin(\beta tx)$ is a solution thus $\dfrac{e^{tx}\sin(\beta tx)}{\beta}$

$$\lim_{\beta\to0}\frac{e^{tx}\sin(\beta tx)}{\beta}=\lim_{\beta\to0}\frac{xe^{tx}\cos(\beta tx)}{1}=xe^{t},$$

If $a=1$ and t is a double root, then the eq'n is:

$$\lambda^2 - 2t\lambda + t^2 = 0 \quad \leftrightsquigarrow \quad y'' - 2ty' + t^2 y = 0$$

Ansatz: $y = xe^{tx}$ $\qquad y' = txe^{tx} + e^{tx}$

$$y'' = t^2xe^{tx} + te^{tx} + te^{tx}$$
$$= t^2xe^{tx} + 2te^{tx}$$

Eqn becomes

$$y'' - 2ty' + t^2y = t^2xe^{tx} + 2te^{tx} - 2t^2xe^{tx}$$
$$- 2te^{tx} + t^2xe^{tx} = 0$$

Better proof: Let $D = \frac{d}{dx}$

then $y'' - 2ty' + t^2y = [\underbrace{D^2 - 2tD + t^2}_{\text{a differential operator}}]y$

$$= (D-t)^2 y$$

$$(D-t)^2(xe^{tx}) = (D-t)\left(e^{tx} + \cancelto{0}{txe^{tx} - txe^{tx}}\right)$$
$$= te^{tx} - te^{tx} = 0$$

Thus: e^{tx} and xe^{tx} are l.i. sol'ns thus the general sol'n, by the thm is

$$Ce^{tx} + Dxe^{tx} \qquad \square$$

$$y'' - y = 0$$

Consider $\lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$, thus the general solution is $Ce^{x} + De^{-x}$

$$y'' - 5y' + 6y = 0$$

Consider $\lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2)$ so the general sol'n is $Ce^{3x} + De^{2x}$

$$y'' + 2y' + 2y = 0$$

Consider $\lambda^2 + 2\lambda + 2 = 0$, $\lambda = \dfrac{-2 \pm \sqrt{4 - 8}}{2}$

$$= -1 \pm i$$

thus the general solution is:

$$Ce^{-x}\cos(x) + De^{-x}\sin(x)$$

$$y'' - 6y' + 9y = 0$$

Consider $\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$

the general solution is $Ce^{3x} + Dxe^{3x}$

Euler equations: $ax^2y'' + bxy' + cy = 0$

$$(xy')' = y' + xy'' \Rightarrow xy'' = (xy')' - y'$$

$$ax^2y'' + bxy' + cy = ax((xy')' - y') + bxy' + cy$$

$$= ax(xy')' + (b-a)xy' + cy$$

$$xy' = x\frac{dy}{dx} = \frac{dy}{d(\log(x))} \qquad \frac{d(\log(x))}{dx} = \frac{1}{x}$$

$$x(xy')' = x\frac{d}{dx}\left(\frac{dy}{d(\log(x))}\right) = \frac{d^2y}{(d(\log(x)))^2} \qquad \Rightarrow d(\log(x)) = \frac{dx}{x}$$

Substituting $t = \log(x)$, the equation becomes

$$ay'' + (b-a)y' + cy = 0$$ (where derivatives are wrt t)

So if $y_1(t)$ is a solution of this eq'n, then $y_1(\log(x))$ is a solution of the original eq'n.

Well if t is a root of $a\lambda^2 + (b-a)\lambda + c$ then e^{tu} shoud solve the above eq'n, so $e^{t\log(x)} = x^t$ should be a sol'n of the original eqn

$$ax^2y'' + bxy' + cy = 0$$

Ansatz: $y = x^t$, then get

$$at(t-1)x^2x^{t-2} + btxx^{t-1} + cx^t$$
$$= (at(t-1) + bt + c)x^t = 0$$

$\Rightarrow$ t is a root of $a\lambda(\lambda-1) + b\lambda + c$
$= a\lambda^2 + (b-a)\lambda + c$

If we have solutions x^{t_1} and x^{t_2}

then $\dfrac{x^{t_1} - x^{t_2}}{t_1 - t_2}$ is a sol'n, so in the limit

$$\lim_{t_1 \to t_2} \frac{x^{t_1} - x^{t_2}}{t_1 - t_2} = \lim_{t_1 \to t_2} \frac{\log(x)\, x^{t_1}}{1} = \log(x)\, x^{t_2}$$

$$x^{t_1} = e^{\log(x) t_1}$$

if $u e^{tu}$ was a sol'n then for $u = \log(x)$ get $\log(x)\, x^{t}$.

If $A e^{\alpha u} \cos(\beta u) + B e^{\alpha u} \sin(\beta u)$ are sol'ns then $A x^{\alpha} \cos(\beta \log(x)) + B x^{\alpha} \sin(\beta \log(x))$ are solutions to the original eqn.

Examples:

$$1 \cdot x^2 y'' + 0 \cdot x y' - 2 \cdot y$$

$$x^2 y'' - 2y = 0$$

Consider $\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$

so the general solution is $C x^2 + D x^{-1}$

$x^2y'' - 3xy' + 4y = 0$

Consider $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$

so the general solution $Cx^2 + Dx^2\log(x)$

$x^2y'' + xy' + 4y = 0$

Consider $\lambda^2 + 4 = (\lambda + 2i)(\lambda - 2i)$ $0 \pm 2i$

so the solutions are $C\cos(2\log(x)) + D\sin(2\log(x))$

$x^2y'' - 3xy' + 5y = 0$

Consider $\lambda^2 - 4\lambda + 5 = 0$ so $\lambda = \frac{4 \pm \sqrt{16-20}}{2}$

$= \frac{4 \pm 2i}{2} = 2 \pm i$

solutions are $Cx^2\cos(\log(x)) + Dx^2\sin(\log(x))$

$(x_1, \dots, x_n)$

Linear Algebra!

A vector space looks like $\mathbb{R}^n$

In $\mathbb{R}^n$, which has as points $v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

a number

If also $w = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ then $v + \lambda w = \begin{bmatrix} x_1 + \lambda y_1 \\ \vdots \\ x_n + \lambda y_n \end{bmatrix}$

let $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ and $e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$] ith posn

↳ elementary vectors

Suppose V is a n-dimensional vector space (elements are lists of n numbers)

suppose that $d : V^n \to \mathbb{R}$

$V^n = \underbrace{V \times \dots \times V}_{\text{n-copies}}$ $d(v_1, \dots, v_n)$ is a number

where i) $d(v_1, \dots, v_i, \dots, v_j, \dots, v_n)$

$= - d(v_1, \dots, v_j, \dots, v_i, \dots, v_n)$

alternating

swapping arguments changes the sign

ii) $d(v_1 + \lambda w_1, v_2, \dots, v_n)$

$= d(v_1, v_2, \dots, v_n) + \lambda d(w_1, v_2, \dots, v_n)$

multilinear

Now consider $d(e_1,\dots,e_n) = \alpha$

These properties and the value α determine d entirely!

Then $d(v_1,\dots,v_n) = \alpha\, |v_1 \dots v_n|$

↳ determinant

The determinant has $|e_1 \dots e_n| = 1$

So the determinant is the unique alternating multilinear function $d: V^n \to \mathbb{R}$ with $d(e_1,\dots,e_n) = 1$

If d has these properties

then $d(v_1, v_2, \dots, v_i + \lambda v_1, \dots, v_n)$

$$= d(v_1,\dots,v_i,\dots,v_n) + \lambda\, \overset{0}{\underbrace{d(v_1,\dots,v_1,\dots,v_n)}_{= -d(v_1,\dots,v_1,\dots,v_n)}}$$

$$= d(v_1,\dots,v_n)$$

$$d(0, v_2, \dots, v_n) = d(v_2, v_2, \dots, v_n) = 0$$

$$d(0+0, v_2, \dots, v_n) = d(0, v_2, \dots, v_n) + d(0, v_2, \dots, v_n)$$

$$\Rightarrow d(0, v_2, \dots, v_n) = 0$$

In a 2D vector space:

$$\left| \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \right| = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = \begin{vmatrix} a & c \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & c \\ b & d \end{vmatrix}$$

$$= \cancelto{0}{\begin{vmatrix} a & c \\ 0 & 0 \end{vmatrix}} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & c \\ b & 0 \end{vmatrix} + \cancelto{0}{\begin{vmatrix} 0 & 0 \\ b & d \end{vmatrix}} = (*)$$

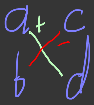
Note $\begin{vmatrix} a & c \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} a & (c - \frac{c}{a}a) \\ 0 & (0 - \frac{c}{a}0) \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & 0 \end{vmatrix} = 0$

$$(*) = ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - bc \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = ad - bc$$

Result $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$

Think



$$\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} \quad \begin{aligned} & x_{11}x_{22}x_{33} - x_{11}x_{23}x_{32} \\ = & + x_{12}x_{23}x_{31} - x_{12}x_{21}x_{33} \\ & + x_{13}x_{21}x_{32} - x_{13}x_{22}x_{31} \end{aligned}$$

The Wronskian

Given n functions $y_1, \dots, y_n$ that have derivatives of order $(n-1)$, consider

$$W[y_1, \dots, y_n](x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & & \cdots & y_n'(x) \\ \vdots & & & \\ y_1^{(n-1)}(x) & & \cdots & y_n^{(n-1)}(x) \end{vmatrix}$$

In particular

$$W[y_1, y_2] = y_1 y_2' - y_1' y_2$$

Suppose we have a 2nd order linear homogeneous ODE:

$$y'' + f(x)y' + g(x)y = 0$$

Suppose this has solutions y_1, y_2

$$W[y_1, y_2] = y_1 y_2' - y_1' y_2$$

$$\frac{d}{dx}\left(W[y_2,y_1]\right) = y_1 y_2'' + y_1' y_2' - y_1'' y_2 - y_1' y_2'$$

$$= y_1 y_2'' - y_1'' y_2 = y_1(-g y_2 - f y_2') - (-g y_1 - f y_1') y_2$$

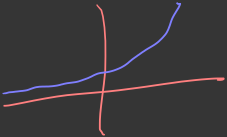
$$= f y_1' y_2 - f y_1 y_2'$$

$$= -f\, W[y_1,y_2]$$

$$\boxed{\begin{array}{c}\text{So } W \text{ satisfies} \\ \dfrac{dW}{dx} = -fW\end{array}}$$

Thus $\dfrac{dW}{W} = -f\,dx \Rightarrow \log(W) = -\int f(x)\,dx$

$$\Rightarrow W = C e^{-\int f(x)dx}$$



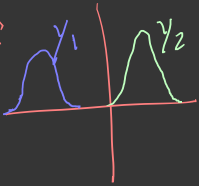
For solutions y_1, y_2, the Wronskian

$W[y_1, y_2]$ is either uniformly zero

or uniformly non-zero

Observation: if y_1, y_2 are linearly dependent, then $W[y_1, y_2] = 0$

Annoying example:



If y_1, y_2 have disjoint support $W[y_1, y_2] = 0$

However: If y_1, y_2 are solutions of $y'' + fy' + gy = 0$ then $W[y_1, y_2] = 0 \Rightarrow y_1$ and y_2 are lin. dep.

Suppose that y_1 is a solution of $y'' + fy' + gy = 0$ then I can find a complimentary solution y_2.

How do I do this?

$$W = y_1 y_2' - y_2 y_1'$$

$$\frac{W}{y_1^2} = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \left(\frac{y_2}{y_1}\right)'$$

$$\frac{y_2}{y_1} = \int \frac{W}{y_1^2}\,dx \Rightarrow y_2 = y_1 \int \frac{W}{y_1^2}\,dx$$

Example: $(x-1)y'' - xy' + y = 0$

Note that $y = e^x$ is a solution

$(\text{as } (x-1)e^x - xe^x + e^x = 0)$

This equation is $y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = 0$

thus $\frac{dW}{W} = \frac{x}{x-1} \Rightarrow \log(W) = \int \frac{x}{x-1}\,dx$

$$= \int \left[1 + \frac{1}{x-1}\right] dx = x + \log(x-1)$$

$$\Rightarrow W = (x-1)e^x$$

Thus we have $y_2 = e^x \int \frac{(x-1)e^x}{e^{2x}}\,dx$

$$= e^x \int (x-1)e^{-x}\,dx \quad \left[\begin{matrix} f = x-1 & dg = e^{-x}dx \\ df = dx & g = -e^{-x} \end{matrix}\right]$$

$$= e^x\left[-(x-1)e^{-x} + \int e^{-x}\,dx\right]$$

$$= e^x\left[-(x-1)e^{-x} - e^{-x}\right] = -xe^{x}e^{-x}$$

$$= -x$$

So the general solutions are $\boxed{Ce^x + Dx}$

Next time: If I have that $\boxed{y_1, y_2}$ are linearly indep. sol'ns of $y''+fy'+gy=0$ then using W and other techniques, I will be able to solve $y''+fy'+gy=h$!