MAT 303: Lecture 3

"$e^{\log(x)x}$"

HW typo: in Q5 b), "x^x" should be "e^x"

Exact first order differential equations:

$$M(x,y)\,dx + N(x,y)\,dy = 0 \iff N y' + M = 0$$

If $K(x,y)$ is some function the total differential of K is $dK = \frac{\partial K}{\partial x} dx + \frac{\partial K}{\partial y} dy$

$\frac{\partial K}{\partial x}$ means diff w.r.t. x, treating y as constant

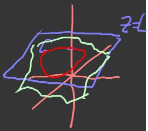
$$K(x,y) = x^3 + 3x^2 y + y^5 + 5$$

$$\frac{\partial K}{\partial x} = K_x = 3x^2 + 6yx$$

$$\frac{\partial K}{\partial y} = K_y = 3x^2 + 5y^4$$

thus $dK = (3x^2 + 6xy)dx + (3x^2 + 5y^4)dy$

Main theorem: The solutions to $dK = 0$ are the curves cut out by the levels of K, i.e. $K(x,y) = C$ for C a constant.



Sketch: Suppose that the curve $K(x,y)=C$ can be parametrised as $K(x,\psi(x))=C$

$$\frac{d}{dx}\left(K(x,\psi(x))\right) = \frac{\partial K}{\partial x}\frac{dx}{dx} + \frac{\partial K}{\partial y}\frac{d\psi}{dx}$$

↳ multi-variable chain rule

$$= \frac{\partial K}{\partial x} + \frac{\partial K}{\partial y}\psi' = \frac{d}{dx}C = 0$$

$\Rightarrow$ Replacing ψ with y, $\frac{\partial K}{\partial x} + \frac{\partial K}{\partial y}\frac{dy}{dx} = 0$

$$\Rightarrow \frac{\partial K}{\partial x}dx + \frac{\partial K}{\partial y}dy = 0 \quad \square$$

When is $M\,dx + N\,dy = 0$ exact?

It is exact when there is K s.t. $dK = N\,dx + M\,dy$

which means $\frac{\partial K}{\partial x} = M \quad \frac{\partial K}{\partial y} = N$

$$\Rightarrow \frac{\partial^2 K}{\partial y\,\partial x} = \frac{\partial M}{\partial y} \qquad \frac{\partial^2 K}{\partial x\,\partial y} = \frac{\partial N}{\partial x}$$

But $\frac{\partial^2 K}{} = \frac{\partial^2 K}{}$ So if $M\,dx + N\,dy$ is exact

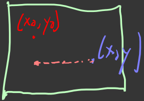
$$\partial y\,\partial x \qquad\qquad \partial x\,\partial y$$

then it follows that $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

If this holds then can we always find K?

Suppose M, N are defined everywhere on $[a,b]\times[c,d]$

we will take all points to live in this rectangle.



Pick (x_0, y_0).

Consider $V(x,y) = \int_{x_0}^{x} M(s,y)\,ds$

$$\frac{\partial V}{\partial x} = M(x,y)$$

if $\underline{dK = M\,dx + N\,dy}$, then

$$\frac{\partial K}{\partial x} = M(x,y) \quad \text{thus} \quad \frac{\partial (K-V)}{\partial x} = 0$$

Thus $K-V$ is a function of y alone

Thus $K = \int_{x_0}^{x} M(s,y)\,ds + \phi(y)$

$$\frac{\partial K}{\partial y} = N = \frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,ds + \phi'(y)$$

$$= \int_{x_0}^{x} \frac{\partial M}{\partial y}(s,y)\,ds + \phi'(y)$$

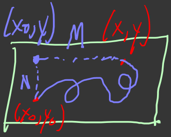
$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$= \int_{x_0}^{x} \frac{\partial N}{\partial x}(s,y)\,ds + \phi'(y)$$

$$= N(x,y) - N(x_0,y) + \phi'(y)$$

$$\Rightarrow N(x,y) = N(x,y) - N(x_0,y) + \phi'(y)$$

$$\Rightarrow \phi'(y) = N(x_0,y)$$



$$\Rightarrow \phi(y) = \int_{y_0}^{y} N(x_0,t)\,dt$$

$$\Rightarrow K(x,y) = \int_{x_0}^{x} M(s,y)\,ds + \int_{y_0}^{y} N(x_0,t)\,dt$$

Remark if M and N fail to be defined, then K might not exist globally.

The the failure of closed forms (i.e. $M_y = N_x$) to be exact ($M\,dx + N\,dy = dK$) is measured by cohomology (studied in graduate algebraic topology courses)

Example: $\underbrace{e^x\cos(2y)}_{M}dx + \underbrace{(9y^2-2e^x\sin(2y))}_{N}dy=0$

Consider $M_y = -2e^x\sin(2y)$

$N_x = -2e^x\sin(2y)$

$M_y = N_x$ so this is exact!

$$K = \int M(x,y)\,dx + \phi(y)$$

$$= \int e^x\cos(2y)\,dx + \phi(y)$$

$$= e^x\cos(2y) + \phi(y)$$

$$N = K_y = -2e^x\sin(2y) + \phi'(y)$$

$$= 9y^2 - 2e^x\sin(2y)$$

$$\Rightarrow \phi'(y) = 9y^2 \Rightarrow \phi(y) = 3y^3$$

$$\Rightarrow K = e^x\cos(2y) + 3y^3$$

$\Rightarrow$ The solutions are $K(x,y) = \boxed{e^x\cos(2y) + 3y^3 = C}$ $\square$

It is very important to check for exactness

e.g., $\underbrace{\cos(x)\cos(y)}_{M}dx + \underbrace{\sin(x)\sin(y)}_{N}dx$

We forgot to check for exactness...

For $dK = Mdx + Ndy$, we have

$$K = \int M(x,y)\,dx + \phi(y)$$

$$= \int \cos(x)\cos(y)\,dx + \phi(y) = \sin(x)\cos(y) + \phi(y)$$

$$K_y = -\sin(x)\sin(y) + \phi'(y) = \sin(x)\sin(y)$$

$$\Rightarrow \phi'(y) = \underbrace{2\sin(x)\sin(y)}_{\text{not a function of } y}$$

likelly does not make sense

Integrating factors:

Remember that the meaning of $Mdx + Ndy = 0$ is that it defines a direction at each point

this is given by slope $-\frac{M}{N}$ (∞ counts as a slope)

If $\mu(x,y)$ is any function then

(1) $\mu M\,dx + \mu N\,dy = 0$ defines the same direction field

If (1) is exact, then μ is called an *integrating factor*

Thm: If $M dx + N dy = 0$ has a solution then it has an integrating factor.

Moreover, it has a solution in all nice cases.

BUT these solutions / integrating factors might not have expressions in elementary functions.

Algorithm for 1st order ODEs is:

Check:
- i) separable? $y' = f(x)g(y)$
- ii) linear? $y' + f(x)y = g(x)$
- iii) Bernoulli? $y' + f(x)y = g(x)y^n$
- iv) homogeneous? $y' = \phi\left(\frac{x}{y}\right)$

If no to all, then:
- v) exact? $M dx + N dy = 0$ with $M_y = N_x$
- vi) find an integrating factor

Know everything here

Well if $\mu M dx + \mu N dy = 0$ is exact, then

$$(\mu M)_y = (\mu N)_x$$

$$\Rightarrow \frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}$$

$$\Rightarrow \boxed{M \frac{\partial \mu}{\partial y} - N \frac{\partial \mu}{\partial x} = \mu \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}$$

equation for an integrating factor

This is a partial differential equation

General solutions to this are outside the scope of ODEs,

However, there are nice cases:

We will assume an ansatz on the form of μ:

Case 1: $\mu(x)$

Case 2: $\mu(y)$

Case 3: $\mu(xy)$

can also consider $\mu(x/y)$

Case 1: $\mu(x)$

$$M \mu_y - N \mu_x = \mu (N_x - M_y)$$

Ansatz $\mu(x,y) = \mu(x)$

$$\Rightarrow -N\mu' = \mu(N_x - M_y)$$

$$\Rightarrow \underbrace{\boxed{\frac{d\mu}{\mu}}}_{\text{function of } x} = \boxed{\frac{N_x - M_y}{-N}}\,dx$$

↳ may or may not be a function of x

If $\frac{N_x - M_y}{-N} \neq f(x)$ then the ansatz was bad!

However, suppose $\frac{N_x - M_y}{-N} = f(x)$, then

$$\Rightarrow \log(\mu) = \int f(x)\,dx$$

$$\Rightarrow \mu = e^{\int f(x)\,dx}$$

Example: $\underbrace{(y + 3y^2 e^{2x})}_{M}dx + \underbrace{(1 + 2ye^{2x})}_{N}dy = 0$

$$\left.\begin{array}{l} M_y = 1 + 6ye^{2x} \\ N_x = 4ye^{2x} \end{array}\right] \text{not exact ☹}$$

However $N_x - M_y = -1 - 2ye^{2x}$

So $\frac{N_x - M_y}{-N} = 1$

$$\Rightarrow \frac{d\mu}{\mu} = 1dx \Rightarrow \log(\mu) = x$$

$$\Rightarrow \mu = e^x$$

So multiplying our starting eq'n by e^x, get:

$$(ye^x + 3y^2e^{3x})dx + (e^x + 2ye^{3x})dy = 0$$

which is exact. If the LHS is dK, then

$$K = \int [ye^x + 3y^2e^{3x}]dx + \phi(y)$$

$$= ye^x + y^2e^{3x} + \phi(y)$$

$$K_y = e^x + 2ye^{3x} + \phi'(y)$$

$$= e^x + 2ye^{3x} \quad \Rightarrow \phi'(y) = 0 \Rightarrow \phi(y) = 0$$

$$\Rightarrow K = ye^x + y^2e^{3x}$$

$\Rightarrow$ The solutions are $\boxed{ye^x + y^2e^{3x} = C}$ $\square$

Case 2: $\mu(y)$

$$M\mu_y - N\mu_x = \mu(N_x - M_y)$$

Ansatz: $\mu(x,y) = \mu(y)$

$$\Rightarrow \quad M\mu' = \mu(N_x - M_y)$$

$$\Rightarrow \quad \frac{d\mu}{\mu} = \boxed{\frac{N_x - M_y}{M}}\, dy$$

$\rightarrow$ the ansatz is only good if

$$\frac{N_x - M_y}{M} = f(y)$$

If that's the case

$$\Rightarrow \quad \log(\mu) = \int f(y)\,dy \Rightarrow \mu = e^{\int f(y)\,dy}$$

Example:

$$\underbrace{(-y\sin(x) + y^3\cos(x))}_{M}dx + \underbrace{(3\cos(x) + 5y^2\sin(x))}_{N}dy = 0$$

$$\left.\begin{aligned} M_y &= -\sin(x) + 3y^2\cos(x) \\ N_x &= -3\sin(x) + 5y^2\cos(x) \end{aligned}\right\} \text{not exact}$$

$$N_x - M_y = -2\sin(x) + 2y^2\cos(x)$$

$$\frac{N_x - M_y}{M} = \frac{2(-\sin(x) + y^2\cos(x))}{y(-\sin(x) + y^2\cos(x))} = \frac{2}{y}$$

Thus we solve

$$\frac{d\mu}{\mu} = \frac{2}{y}dy \Rightarrow \log(\mu) = 2\log(y)$$

$$\Rightarrow \mu = y^2$$

Thus multiplying the original eq'n by y^2, get:

$$(-y^3\sin(x) + y^5\cos(x))dx + (3y^2\cos(x) + 5y^4\sin(x))dx = 0$$

which is exact.

For $LHS = dK$, have

$$K = \int[-y^3\sin(x) + y^5\cos(x)]dx + \phi(y)$$

$$= y^3\cos(x) + y^5\sin(x) + \phi(y)$$

$$K_y = 3y^2\cos(x) + 5y^4\sin(x) + \phi'(y)$$

$$= 3y^2\cos(x) + 5y^4\sin(x)$$

$$\Rightarrow \phi' = 0 \Rightarrow \phi = 0$$

$$\Rightarrow K = y^3\cos(x) + y^5\sin(x)$$

$\Rightarrow$ Solutions are $y^3\cos(x) + y^5\sin(x) = C$

$$\frac{d}{dy}\mu(2y) = 2\mu'(2y)$$

Case 3: $\mu(xy)$

$$M\mu_y - N\mu_x = \mu(N_x - M_y)$$

Ansatz $\mu(x,y) = \mu(xy)$

$$\Rightarrow xM\mu'(xy) - yN\mu'(xy) = \mu(xy)(N_x - M_y)$$

$z = xy$

$$\Rightarrow \frac{d\mu}{\mu} = \boxed{\frac{N_x - M_y}{xM - yN}}\, dz$$

$\rightarrow$ Ansatz is useful iff this is a function of $z = xy$

Example:

$$\Big(\underbrace{3y\cos(x+y) - xy\sin(x+y)}_{M}\Big)dx + \Big(\underbrace{3x\cos(x+y) - xy\sin(x+y)}_{N}\Big)dy = 0$$

$$\left.\begin{aligned}
M_y &= 3\cos(x+y) - 3y\sin(x+y) - x\sin(x+y) - xy\cos(x+y) \\
&= (3-xy)\cos(x+y) - (3y+x)\sin(x+y) \\
N_x &= 3\cos(x+y) - 3x\sin(x+y) - y\sin(x+y) - xy\cos(x+y) \\
&= (3-xy)\cos(x+y) - (3x+y)\sin(x+y)
\end{aligned}\right\}\text{not exact}$$

$$\begin{aligned}
N_x - M_y &= (-3x - y + 3y + x)\sin(x+y) \\
&= 2(y-x)\sin(x+y)
\end{aligned}$$

$$xM - yN = \left(-x^2y + xy^2\right)\sin(x+y)$$

$$= xy\,(y-x)\,\sin(x+y)$$

$$\frac{N_x - M_y}{xM - yN} = \frac{2(y-x)\sin(x+y)}{xy(y-x)\sin(x+y)} = \frac{2}{xy} = f(xy)$$

for $f(z) = \frac{2}{z}$

Then consider $\frac{d\mu}{\mu} = f(z)dz = \frac{2}{z}dz$

$$\Rightarrow \log(\mu) = 2\log(z)$$

$$\Rightarrow \mu = z^2 = x^2y^2$$

Then mutiplying the original eq'n by x^2y^2 get:

$$\left(3x^2y^3\cos(x+y) - x^3y^3\sin(x+y)\right)dx$$
$$+ \left(3x^3y^2\cos(x+y) - x^3y^3\sin(x+y)\right)dy = 0$$

Which is exact. Thus for $LHS = dK$, get

$$K = \int \left[3x^2y^3\cos(x+y) - x^3y^3\sin(x+y)\right]dx + \phi(y)$$

(by product rule)
$$= x^3y^3\cos(x+y) + \phi(y)$$

$$K_y = 3x^3y^2\cos(x+y) - x^3y^3\sin(x+y) + \phi'(y)$$

$$= 3x^3y^2\cos(x+y) - x^3y^3\sin(x+y)$$

$$\Rightarrow \phi' = 0 \Rightarrow \phi = 0$$

$$K = x^3y^3\cos(x+y)$$

$\Rightarrow$ The solutions are $\boxed{x^3y^3\cos(x+y) = C}$ ☒

End of first order theory!!

Second order ODEs

In general a second order ODE is an equation of the form

$$F(x, y, y', y'') = 0$$

A linear 2nd order ODE has the form:

$$y'' + f(x)y' + g(x)y = h(x)$$

This is homogeneous when $h(x) = 0$

A special case of this is when f, g are numbers

A general 2nd order linear ODE with constant coefficients has the form:

$$ay'' + by' + cy = h(x) \qquad \text{for } a,b,c \ \underline{\text{numbers}}$$

Let's solve the homogeneous equation:

$$ay'' + by' + cy = 0$$

Ansatz: $y = e^{tx}$

$$at^2e^{tx} + bte^{tx} + ce^{tx} = 0$$

$$(at^2 + bt + c)e^{tx} = 0$$

Thus t is a solution of the quadratic

$$a\lambda^2 + b\lambda + c = 0$$

Quadratic eq'n review:

solutions are: $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x^2 + 2x + 1 = (x+1)^2$$

$$x^2 - 1 = (x-1)(x+1)$$

$$\boxed{x^2 + 1} = x^2 - i^2 \qquad \frac{\pm\sqrt{-4}}{2} = \pm\sqrt{-1} = \pm i$$

$$= (x-i)(x+i)$$

$$i^2 = -1$$

Complex numbers are numbers of the form

$x + iy$ for $x, y \in \mathbb{R}$

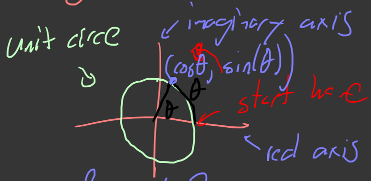
$$(a+ib)(c+id) = ac + aid + ibc + i^2bd$$

$$= (ac - bd) + (ad + bc)i$$

→ a real number

Euler's formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$$e^{\alpha+i\beta} = e^{\alpha} e^{i\beta} = e^{\alpha}(\cos(\beta)+\sin(\beta) i)$$



radians are a measure of arclength

circumference: 2π

$z \in \mathbb{C}$ then $z = re^{i\theta}$ is the polar representation

$$\lambda^2+1=0$$

this corresponds to the differential equation

$y''+y=0$ solutions are e^{it} e^{-it}

If $a\lambda^2+b\lambda+c=0$ will have complex solution

the discriminant $b^2-4ac<0$

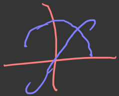
in this case the solutions are

going to be $\underline{\alpha+i\beta}$ $\underline{\alpha-i\beta}$

conjugate pair

$\alpha = \frac{-b}{2a}$ $\beta = \frac{\sqrt{4ac-b^2}}{2a}$

In $y''+y=0$ got that the solutions are.



$$y = Ce^{it} + De^{-it}$$

$$= C(\cos(t) + i\sin(t)) + D(\cos(-t) + i\sin(-t))$$

$$= (C+D)\cos(t) + i(C-D)\sin(t)$$

$C+D=\alpha \quad i(C-B)=\beta$

$$= \underline{\alpha\cos(t) + \beta\sin(t)}$$