MAT 303: Lecture 2

1st Order Linear ODEs:

(1) $y' + f(x)y = g(x)$

(2) $y' + f(x)y = 0$

Solving the homogeneous problem:

$$\frac{dy}{dx} = -f(x)y \Rightarrow \frac{dy}{y} = -f(x)dx$$

$$\Rightarrow \log(y) = -\int f(x)dx + \log(C)$$

$$\Rightarrow y = Ce^{-F(x)} \quad \text{where} \quad F(x) = \int f(x)dx$$

Solving the general problem:

Method I: Integrating Factors

Multiply (1) by $\mu(x)$ ← some f'n of x

$$\mu(x)y' + \mu(x)f(x)y = g(x)\mu(x)$$

Suppose that the LHS had the form $(\mu(x)y)'$

then we would have $(\mu(x)y)' = g(x)\mu(x)$

$$\Rightarrow \mu(x)\, y = \int g(x)\mu(x)\,dx + C \quad (*)$$

Well, to get this we want $(\mu(x)y)' = \mu(x)y' + \mu'(x)y$

$$= \mu(x)y' + \mu(x)f(x)y, \text{ so } \mu'(x) = \mu(x)f(x)$$

$$\Rightarrow \mu' - f(x)\mu = 0$$ this is a homogeneous equation!

Then a particular solution is $\mu(x) = e^{F(x)}$

So going back to our equation $(*)$ we get

$$y = \mu^{-1}\int g(x)\mu(x)\,dx + C\mu^{-1}$$

$$\Rightarrow \boxed{y = e^{-F(x)}\int g(x)e^{F(x)}\,dx + Ce^{-F(x)}}$$

Examples:

$$y' + 3x^{-1}y = 5x$$

Soln: We will first find an integrating factor:

$$\mu' = 3x^{-1}\mu \Rightarrow \frac{d\mu}{\mu} = \frac{3\,dx}{x} \Rightarrow \log(\mu) = 3\log(x)$$

$\Rightarrow \mu = x^3$

We multiply the equation by x^3 to obtain:

$$(x^3y)' = x^3y' + 3x^2y = 5x^4$$

$$\Rightarrow x^3y = \int 5x^4\,dx = x^5 + C$$

↱ sol'n to homogeneous eq'n

$$\Rightarrow y = x^2 + Cx^{-3} \qquad \square$$

↳ particular sol'n

$$y' + \tan(x)y = \cos(x)$$

First we find an integrating factor:

$$\mu' = \tan(x)\mu \Rightarrow \frac{d\mu}{\mu} = \tan(x)dx$$

$$\Rightarrow \log(\mu) = \int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx \quad \left[\begin{array}{l}\text{Let } u = \cos(x)\\ du = -\sin(x)dx\end{array}\right]$$

$$= -\int \frac{du}{u} = -\log(u) = -\log(\cos(x)) \Rightarrow \mu = \frac{1}{\cos(x)}$$

Multiply by $\frac{1}{\cos(x)}$ to get $\left(\frac{y}{\cos(x)}\right)' = \frac{y'}{\cos(x)} + \frac{\sin(x)}{\cos^2(x)}y = 1$

$$\Rightarrow \frac{y}{\cos(x)} = \int dx = x + C$$

$$\Rightarrow y = x\cos(x) + C\cos(x) \quad \square$$

Method II : Variation of Parameters

Suppose y_1 is a solution of the homogeneous eq'n

i.e., $y_1' + f(x)y_1 = 0$

Then the general solution to the homogeneous problem is $y = Cy_1$

Ansatz (guess as to the form of the final solution):

suppose the solution to the general equation is $y = uy_1$

where $u = u(x)$ is some function of x

Then we have that $(uy_1)' + fuy_1 = g$

$$\Rightarrow u'y_1 + uy_1' + fuy_1 = g$$

$$\Rightarrow u'y_1 + u\overset{0}{\cancel{(y_1' + fy_1)}} = g$$

$$\Rightarrow u'y_1 = g \quad \Rightarrow \quad u' = g\,y_1^{-1}$$

$$\Rightarrow u(x) = \int g(x)\,y_1^{-1}(x)\,dx + C$$

$(y = uy_1)$

$$\Rightarrow y = \underbrace{y_1 \int g(x)\, y_1^{-1}(x)\, dx}_{\text{particular sol'n}} + \underbrace{Cy_1}_{\text{sol'n of homogeneous eq'n}}$$

Example: $y' - \frac{t}{t^2+1} y = t$

Using the method of variation. First we solve the homogeneous eq'n:

$$y' = \frac{t}{t^2+1} y \Rightarrow \frac{dy_1}{y} = \frac{t}{t^2+1} dt$$

$$\Rightarrow \log(y) = \int \frac{t}{t^2+1} dt = \frac{1}{2} \log(t^2+1)$$

$$\Rightarrow y = \sqrt{t^2+1}$$

We will use the ansatz $y = u\sqrt{t^2+1}$, thus

$$\left(u\sqrt{t^2+1}\right)' - \frac{t}{t^2+1} u \sqrt{t^2+1} = t$$

$$\Rightarrow u' \sqrt{t^2+1} + \cancelto{0}{u \frac{t}{\sqrt{t^2+1}} - \frac{t}{\sqrt{t^2+1}} u} = t$$

$$\Rightarrow u' = \frac{t}{\sqrt{t^2+1}} \Rightarrow u = \int \frac{t}{\sqrt{t^2+1}} dt = \sqrt{t^2+1} + C$$

$$\Rightarrow y = u\sqrt{t^2+1} = \underbrace{t^2+1}_{\text{particular sol'n}} + \overbrace{C\sqrt{t^2+1}}^{\text{sol'n to homogeneous}} \quad \square$$

Special First Order ODEs

$$y' = f(x,y)$$

Bernoulli's Equations: $y' + f(x)y = g(x)y^n$ for $n \geq 2$

This is not linear, but it can be reduced to one!

$$(y^{1-n})' = (1-n)y^{-n}y'$$

$$y^{-n}y' + f(x)y^{1-n} = g(x)$$

$$\Rightarrow \frac{1}{1-n}(y^{1-n})' + f(x)y^{1-n} = g(x)$$

Let $z = y^{1-n}$, then the equation becomes

$$\frac{1}{1-n}z' + f(x)z = g(x)$$ first order linear equation!

Example! $y' + y = y^2$

This is a Bernoulli equation with $n = 2$ so let $z = y^{1-2} = y^{-1}$

Thus $y = z^{-1}$, and $(z^{-1})' + z^{-1} = z^{-2}$

$$\Rightarrow \frac{-z'}{z^2} + \frac{1}{z} = \frac{1}{z^2} \Rightarrow \boxed{z' - z = -1}$$

Now solving this linear equation, by variation, we first solve the homogeneous problem.

$$z' = z \Rightarrow \frac{dz}{z} = dx \Rightarrow \log(z) = x$$

$\Rightarrow z = e^x$

We use the ansatz $z = ue^x$, thus

$$(ue^x)' - ue^x = -1$$

$$\Rightarrow u'e^x \underbrace{+ ue^x - ue^x}_{\to 0} = -1 \quad \Rightarrow u' = -e^{-x}$$

$$\Rightarrow u = -\int e^{-x}\,dx = e^{-x} + C$$

$$\Rightarrow z = 1 + Ce^x \quad \Rightarrow y = \frac{1}{1 + Ce^x}$$

Second method: Just because you can reduce to a linear equation does not mean that you should.

$$y' + f(x)y = g(x)y^n \qquad n \geq 2$$

Let y_1 be a solution of $y' + f(x)y = 0$

Ansatz: $y = uy_1$

$$(uy_1)' + fuy_1 = gu^n y_1^n$$

$$\left(e^{-F(x)}\right)' = -f(x)\,e^{-F(x)}$$

$$\Rightarrow u' y_1 + u y_1' + f u y_1 = g u^n y_1^n$$

by FTC-II

We have $y_1 = e^{-F(x)}$ where $F(x) = \int f(x)\,dx$

$$\Rightarrow u' e^{-F(x)} \cancelto{0}{- u f(x) e^{-F(x)} + f(x) u e^{-F(x)}} = g u^n e^{-nF(x)}$$

$$\Rightarrow u' = g(x) u^n e^{-(n-1)F(x)}$$

$$\left(\frac{u^{1-n}}{1-n}\right)' = u^{-n}$$

$$\Rightarrow \frac{du}{u^n} = g(x) e^{-(n-1)F(x)} dx$$

$$\Rightarrow \frac{1}{(1-n)u^{n-1}} = \int g(x) e^{-(n-1)F(x)} dx$$

$$\Rightarrow u = \left((1-n)\int g(x) e^{-(n-1)F(x)} dx\right)^{\frac{1}{1-n}}$$

$$\Rightarrow y = e^{-F(x)}\left((1-n)\int g(x) e^{-(n-1)F(x)} dx\right)^{\frac{1}{1-n}}$$

Example $y' + y = y^2$

First we solve the homogeneous problem

$$y' = -y \Rightarrow \frac{dy}{y} = -dx \Rightarrow \log(y) = -x$$

$\Rightarrow y = e^{-x}$ $\qquad e^{x^2} := e^{(x^2)}$

Ansatz: $y = ue^{-x}$, thus $\qquad (e^x)^2 = e^{2x}$

$$(ue^{-x})' + ue^{-x} = u^2 e^{-2x}$$

$$\Rightarrow u'e^{-x} \overset{0}{\cancel{- ue^{-x} + ue^{-x}}} = u^2 e^{-2x}$$

$$\Rightarrow u' = u^2 e^{-x} \Rightarrow \frac{du}{u^2} = e^{-x} dx$$

$$\Rightarrow \frac{-1}{u} = \int e^{-x} dx = -e^{-x} + C$$

$$\Rightarrow u = \frac{1}{e^{-x} - C} \Rightarrow y = ue^{-x} = \frac{1}{1 - Ce^{x}} \qquad \square$$

Homogeneous Equation (distinct from homogeneous linear)

$$\frac{dy}{dx} = f(x,y) = \varphi\left(\frac{y}{x}\right)$$

Ansatz: $y = ux \Rightarrow u = \frac{y}{x}$

Get: $(ux)' = \varphi(u)$

$$\Rightarrow u'x + u = \varphi(u)$$

$$\Rightarrow u' = \frac{\varphi(u) - u}{x}$$

$$\Rightarrow \frac{du}{\varphi(u)-u} = \frac{dx}{x} \Rightarrow \log(x) = \int \frac{du}{\varphi(u)-u} + \log(C)$$

$$\Rightarrow \begin{cases} x = C\, e^{\int \frac{du}{\varphi(u)-u}} \\ y = ux \end{cases}$$ parametric solution in terms of u

Example: $$y' = \frac{x^2+y^2}{x^2+xy} = \frac{1+(y/x)^2}{1+y/x}$$

Ansatz $u = \frac{y}{x}$ i.e. $y = ux$

$$(ux)' = \frac{1+u^2}{1+u}$$

$$\Rightarrow u'x + u = \frac{1+u^2}{1+u} \Rightarrow u'x = \frac{1+u^2}{1+u} - u$$

$$= \frac{1+u^2 - u(1+u)}{1+u}$$

$$= \frac{1-u}{1+u}$$

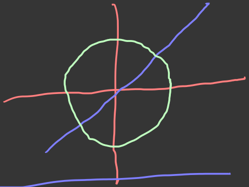
$$\Rightarrow \frac{1+u}{1-u}\, du = \frac{dx}{x}$$

$$\Rightarrow \log(x) = \int \frac{1+u}{1-u}\,du = \int \frac{-(1-u)+2}{1-u}\,du$$

$$= \int \left[1 - \frac{2}{u-1}\right] du = -u - 2\log(u-1) + \log(C)$$

$$\Rightarrow x = C\,e^{-u-2\log(u-1)} = C\,e^{-u}\,e^{-2\log(u-1)}$$

$$\Rightarrow \begin{cases} x = \dfrac{C}{e^{u}(u-1)^2} \\ y = ux \end{cases} \qquad \square$$



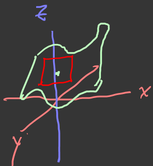
Exact Differential Equations

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} \Rightarrow f(x,y)dx - g(x,y)dy = 0$$

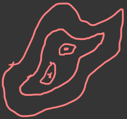
Consider instead $\boxed{f(x,y)dx + g(x,y)dy = 0}$

$K(x,y)$ is a function

$$dK = \frac{\partial K}{\partial x}dx + \frac{\partial K}{\partial y}dy$$



suppose $f(x,y)\,dx + g(x,y)\,dy = dK$

then we have $K(x,y) = C$ are the general solutions!