MAT 303 Lecture 15 (Numerics)

⚠ Most of the first half was done in Racket

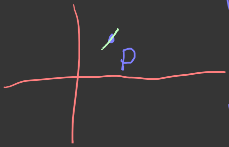
First order ODE:

$y' = f(x,y)$

this defines a direction field, and we want integral curves of this

Let $P_0 = (x_0, y_0)$ lets try and find a curve through this point.

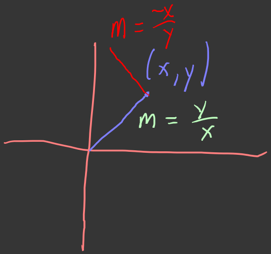
$\Delta(x,y) = (1, f(x,y))$

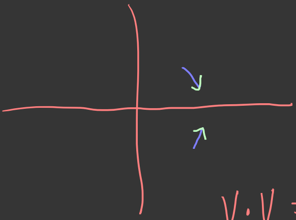


Ver 1: $P_1 = P_0 + h\,\Delta(x,y)$

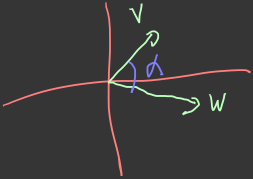
Ver 2: $P_1 = P_0 + h \dfrac{\Delta(x,y)}{|\Delta(x,y)|}$

Δ is a vector in the plane, so $|\Delta| = \sqrt{\Delta_x^2 + \Delta_y^2}$





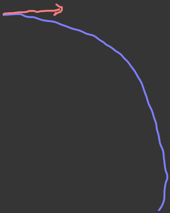
$V \cdot V = V_x^2 + V_y^2$



$$= |v|^2$$

$$V\cdot W = V_x W_x + V_y W_y$$

$$\boxed{\frac{V\cdot W}{|v|\,|w|} = \cos(\alpha)}$$



Population dynamics:

$$y' = y(1-y)$$

$$y' = f(x,y) \iff f(x,y)dx - dy = 0$$

$$\underbrace{(f(x,y),-1)}_{\hookrightarrow \text{ orthogonal to } (1,\, f(x,y))}\cdot(dx,dy)$$

$$Mdx + Ndy = 0$$

$$\iff \mu Mdx + \mu Ndy = 0$$

$$y' = \frac{-x}{y} \Leftrightarrow y\,dy = -x\,dx$$

$$\Leftrightarrow x\,dx + y\,dy = 0$$

(dx, dy) is orthogonal to (x, y)

Second order theory

$$y'' - y = \frac{2}{e^t + 1} \qquad \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

Parameters: $t \quad y \quad y'$ $\qquad y'' = y + \frac{2}{e^t+1}$

$$P_0 = (t_0, y_0, y'_0) \qquad \delta_t$$

$$P_1 = \left(t_0 + \delta_t,\ y_0 + \delta_t y'_0,\ y'_0 + \delta_t\left(y_0 + \frac{2}{e^{t_0}+1}\right)\right)$$

Generic second order problem:

$$\boxed{y'' = f(t, y, y')}$$

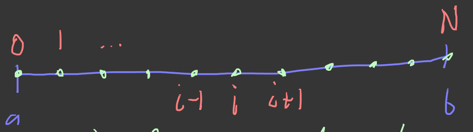
Everything we did before is known as a
finite difference method

You commit to steps and the state is small
(e.g. (t_n, y_n, y'_n) - three numbers in the second order case)

Instead we can take a huge state and iterate on that

Idea: descritise space and store a discrete function

$$f(t) \rightsquigarrow f[i]$$



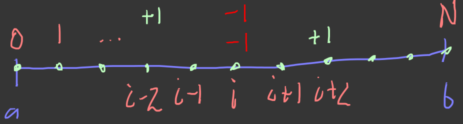
(N+1) uniformly spaced points

the the distance between points is $\frac{b-a}{N} =: \boxed{h}$

(Finite element method)

Discrete derivatives: $f'[i] = \frac{f[i+1] - f[i-1]}{2h}$

What about higher derivatives?



$$f''[i] = \frac{f'[i+1] - f'[i-1]}{2h}$$

$$= \frac{f[i+2] + f[i-2] - 2f[i]}{4h^2}$$

Linear first order equation:

$$\underbrace{y' + f(t)y}_{} = g(t)$$

f g also discrete

$f[i]$ $g[i]$

$$\mathcal{D} = D + f$$

$$\mathcal{D}[y] = g(t)$$

$$y'[i] + f[i] \cdot y[i] = g[i]$$

$$\frac{1}{2h}\left(y[i+1]-y[i-1]\right)+f[i]\cdot y[i]=g[i]$$

This is a linear equation in $\mathbb{R}^{N+1}$

$$\begin{array}{c|ccccc}
A & y[i-2] & y[i-1] & y[i] & y[i+1] & y[i+2] \\
\hline
y[i-3] & f[i-2] & \frac{1}{2h} & & 0 & \\
y[i-2] & -\frac{1}{2h} & f[i-1] & \frac{1}{2h} & & \\
y[i] & & -\frac{1}{2h} & f[i] & \frac{1}{2h} & \\
y[i+2] & 0 & & -\frac{1}{2h} & f[i+1] & \frac{1}{2h} \\
y[i+3] & & & & -\frac{1}{2h} & f[i+2]
\end{array}$$

Solving $A\vec{y}=\vec{g}$ A is a $(N+1)\times(N+1)$ matrix

We have reduced this to a linear algebra problem, with a huge matrix whose entries cluster around the diagonal.

Are there good ways to solve problems like this?

Method (Gauss - Sidele)

$A = D + F$ where D is diagonal and F has no diagonal entries

If you want to solve $A\vec{x} = \vec{b}$

then we have $A\vec{x} = D\vec{x} + F\vec{x} = \vec{b}$

$$\Rightarrow D\vec{x} = \vec{b} - F\vec{x} \Rightarrow \vec{x} = D^{-1}(\vec{b} - F\vec{x})$$

Turn this into an iterative method:

Start with $\vec{x}_0 = \vec{0}$

Update via $\vec{x}_{n+1} = D^{-1}(\vec{b} - F\vec{x}_n)$

This is nonsense if any diagonal entries are zero

Can do linear algebra analysis with the spectral radius to prove conditional convergence theorems.

$$f : \mathbb{R}^3 \to \mathbb{R}$$

$$\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

In 1D the laplacian is D^2

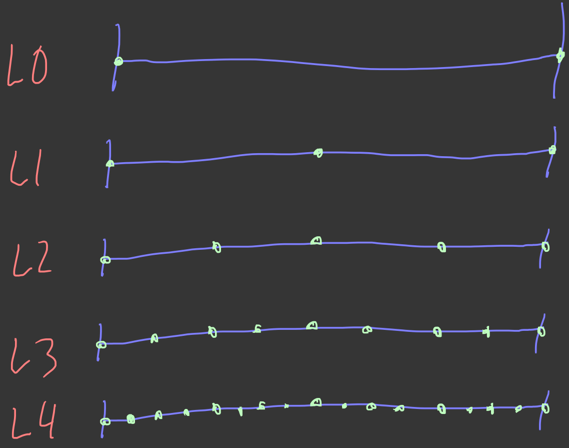
High brained thing: Do fourier analysis

the convergence of this process is slowed down by low frequency modes

Interactions at a distance are problematic,

Multigrid methods

Work on multiple levels of discretisatio



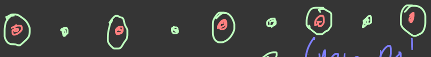
Start solving at L0 (in practice do N=50)

At N=50 G-S will converge in ~10 iterations

This handles interactions at a distance quickly

At this point you refine your solution

$f^e[i-1]$ $f^e[i]$ $f^e[i+1]$



(new points)

assign this the average of the points around it

Start itterating at L2 with this lifted solution

In practice this will take equally many itterations

Repeat until you get to the top

L4	10	160	1024
L3	10		512
			256
L2	10		128
L1	10		
L0	10		64

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} < 2$$