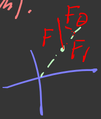
MAT 303 Lecture 14

Central motion (continued)

Theorem 2 (conservation of angular momentum):



In polar coordinates we $\vec{F} = (F_r, F_\theta)$

$F_\theta = 0$ for a central force

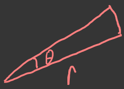
$$F_\theta = 0 = m a_\theta = m(2r'\theta' + r\theta'')$$

$$\Rightarrow 0 = 2rr'\theta' + r^2\theta'' = (r^2\theta')'$$

$$\Rightarrow \quad r^2\theta' = h$$

(some constant, the angular momentum of the system) ∎

Theorem 3 (equal areas are swept out in equal times):



$$A = \pi r^2 \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta$$

$$dA = \frac{1}{2} r^2 d\theta \Rightarrow A' = \frac{1}{2} r^2 \theta' = \frac{1}{2} h$$

$$\Rightarrow A = \frac{1}{2} h t$$

(area increases linearly) ∎

Force fields

If you have a force at every point this is $\vec{F}(\vec{r})$

Consider $F_x dx + F_y dy + F_z dz$

We can ask if this is dU for some scalar function U; this would mean

$$\frac{\partial U}{\partial x} = F_x \qquad \frac{\partial U}{\partial y} = F_y \qquad \frac{\partial U}{\partial z} = F_z$$

If such a U exists, then $-U$ is the potential of the field F, the field is then known as conservative

Example 1: $F = -kr$ (spring)

$$F_x = \frac{x}{r}(-kr) = -kx \qquad F_y = \frac{y}{r}(-kr) = -ky$$

$$F_z = \frac{z}{r}(-kr) = -kz$$

$$U(x,y,z) = \frac{-k}{2}\left(x^2+y^2+z^2\right) = \frac{-k}{2}r^2$$

Example 2: $F = \dfrac{-k}{r^2}$ $\qquad r = \sqrt{x^2+y^2+z^2}$

$$F_x = \frac{x}{r}\left(\frac{-k}{r^2}\right) = \frac{-kx}{r^3} \qquad F_y = \frac{-ky}{r^3} \qquad F_z = \frac{-kz}{r^3}$$

$$U(x,y,z) = \frac{k}{\sqrt{x^2+y^2+z^2}} = \frac{k}{r}$$

Theorem 4 (in a conservative field, energy is conserved)

$$dU = F_x\,dx + F_y\,dy + F_z\,dz$$

$$\Rightarrow \frac{\partial U}{\partial x} = m x'', \quad \frac{\partial U}{\partial y} = m y'', \quad \frac{\partial U}{\partial z} = m z''$$

$$m x'' \frac{dx}{dt} + m y'' \frac{dy}{dt} + m z'' \frac{dz}{dt} = \frac{dU}{dt}$$

$$\Rightarrow \frac{1}{2} m v_x^2 + \frac{1}{2} m v_y^2 + \frac{1}{2} m v_z^2 = U + E$$

$$\Rightarrow \frac{1}{2} m |\vec{v}|^2 - U = E \quad \text{(constant)} \qquad \square$$

Motion when $F = -k^2 m r$ (movment in xy plane)

$$F_x = m x'' = F \frac{x}{r} = -k^2 m x$$

$$F_y = m y'' = -k^2 m y$$

$$\begin{cases} x'' = -k^2 x \\ y'' = -k^2 y \end{cases}$$

$$\lambda^2 + k^2 = (\lambda + ki)(\lambda - ki)$$

$$x = A\cos(kt) + B\sin(kt)$$

$$y = C\cos(kt) + D\sin(kt)$$

$$\Rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \cos(kt) \\ \sin(kt) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos(kt) \\ \sin(kt) \end{bmatrix} = \frac{1}{AD-BC} \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \frac{1}{AD-BC}\begin{bmatrix} Dx - By \\ -Cx + Ay \end{bmatrix}$$

$$1 = \cos^2(kt) + \sin^2(kt)$$

$$= \frac{(Dx - By)^2 + (-Cx + Ay)^2}{(AD-BC)^2}$$

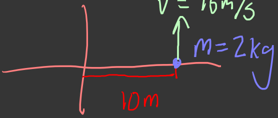
$$\Rightarrow (AD-BC)^2 =$$

$$(D^2 + C^2)x^2 + -2(BD + AC)xy + (B^2 + A^2)y^2$$

$$\alpha^2 x^2 + \beta xy + \gamma^2 y^2 = \delta$$ (equation of an ellipse)

Example

$$F = -\frac{1}{8} r \qquad 2x'' = -\frac{x}{8}$$



$$2y'' = \frac{-y}{8}$$

$x = A\cos\left(\frac{t}{4}\right) + B\sin\left(\frac{t}{4}\right)$

$y = C\cos\left(\frac{t}{4}\right) + D\sin\left(\frac{t}{4}\right)$

$x' = \frac{-1}{4}A\sin\left(\frac{t}{4}\right) + \frac{1}{4}B\cos\left(\frac{t}{4}\right)$

$y' = \frac{-1}{4}C\sin\left(\frac{t}{4}\right) + \frac{1}{4}D\cos\left(\frac{t}{4}\right)$

1) $x(0) = 10$

2) $y(0) = 0$

3) $x'(0) = 0$

4) $y'(0) = 15$

1) $10 = A$ 2) $0 = C$

3) $0 = \frac{1}{4}B$ 4) $15 = \frac{1}{4}D$

$x = 10\cos\left(\frac{t}{4}\right)$

$y = 60\sin\left(\frac{t}{4}\right)$

$\Rightarrow 1 = \left(\frac{x}{10}\right)^2 + \left(\frac{y}{60}\right)^2$

Motion when $F = -\frac{Km}{r^2}$ (again in the plane)

$F_x = \frac{x}{r}\left(-\frac{Km}{r^2}\right) = \frac{-Kmx}{r^3} = mx'' \Rightarrow x'' = \frac{-kx}{(x^2+y^2)^{3/2}}$

We do not get decoupled equations!

Do this in polar coordinates!

$$F_r = ma_r = m(r'' - r(\theta')^2)$$

$$F_\theta = ma_\theta = m(2r'\theta' + r\theta'')$$

a constant in our solution

$$F_\theta = 0 \text{ (because central)} \Rightarrow r^2\theta' = h \Rightarrow \theta' = \frac{h}{r^2}$$

$$F_r = \frac{-Km}{r^2} = m\left(r'' - r(\theta')^2\right) = m\left(r'' - \frac{h^2}{r^3}\right)$$

$$\Rightarrow r'' - \frac{h^2}{r^3} = \frac{-K}{r^2}$$

Goal: solve for r as a function of θ

$$u = \frac{1}{r} \qquad \theta' = hu^2 \qquad r = \frac{1}{u}$$

Physicist proof:

$$r' = \frac{-1}{u^2}\frac{du}{dt} = \frac{-1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}$$

$$r'' = -h \frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}$$

Translation:

$$\tilde{u} = \tilde{u}(\theta)$$

Ansatz: $u(t) = \tilde{u}(\theta(t))$

$$r' = \frac{-1}{u^2}\frac{du}{dt} = \frac{-1}{u^2}\frac{d}{dt}\left(\tilde{u}(\theta(t))\right) = \frac{-1}{u^2}\frac{d\tilde{u}}{d\theta}(\theta(t))\frac{d\theta}{dt}$$

$$= -h\frac{d\tilde{u}}{d\theta}(\theta(t))$$

$$r'' = -h\frac{d^2\tilde{u}}{d\theta^2}(\theta(t))\frac{d\theta}{dt} = -h^2u^2\frac{d^2\tilde{u}}{d\theta^2}(\theta(t))$$

Recall:

$$r'' - \frac{h^2}{r^3} = \frac{-K}{r^2} \qquad u = \frac{1}{r}$$

$$-h^2u^2\frac{d^2\tilde{u}}{d\theta^2}(\theta(t)) - h^2u^3 = -Ku^2$$

$$\frac{d^2\tilde{u}}{d\theta^2}(\theta(t)) + \tilde{u}(\theta(t)) = \frac{K}{h^2}$$

Abstract this to replace $\theta(t)$ with an arbitrary θ

$$\tilde{u}'' + \tilde{u} = \frac{k}{h^2}$$ (second order linear ODE for $\tilde{u}$)

homogeneous eqn: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

$\tilde{u} = A\cos(\theta - \delta)$ sol'n to homogeneous

$\tilde{u} = \frac{k}{h^2}$ solves the inhomogeneous problem (by inspection)

$$\tilde{u} = A\cos(\theta - \delta) + \frac{k}{h^2}$$

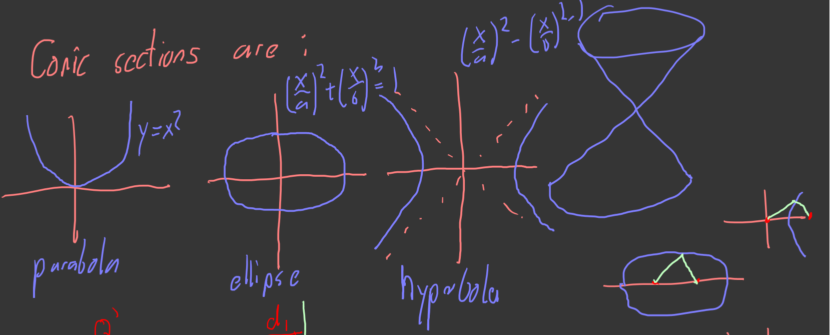
Let $A = e\frac{k}{h^2}$, then

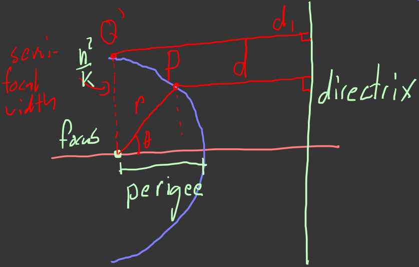
$$\tilde{u}(\theta) = \frac{k}{h^2}\left(1 + e\cos(\theta - \delta)\right) \Rightarrow \tilde{r}(\theta) = \frac{h^2}{k}\,\frac{1}{e\cos(\theta\underset{-\delta}{\wedge})}$$

Choose coordinates such that $\delta = 0$

Goal: show this is the equation of a conic section

Conic sections are :


$(\frac{x}{a})^2 - (\frac{x}{b})^2 = 1$
$(\frac{x}{a})^2 + (\frac{x}{b})^2 = 1$
$y = x^2$
parabola
ellipse
hyperbola



$\frac{r}{d} = e$ is constant

e is the eccentricity of the conic section

$$e = \frac{r}{d} \qquad e = \frac{h^2}{K d_1} \Rightarrow d_1 = \frac{h^2}{Ke}$$

$$d = \frac{r}{e}$$

$$d_1 = r\cos\theta + d \Rightarrow \frac{h^2}{Ke} = \frac{r}{e} + r\cos\theta$$

$$\Rightarrow \frac{h^2}{K} = r(1 + e\cos(\theta))$$

$$\Rightarrow r = \frac{h^2}{K} \frac{1}{1 + e\cos(\theta)}$$

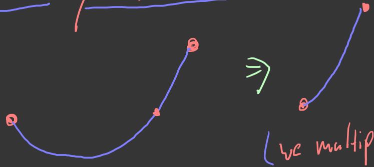
$e < 1 \Rightarrow$ elliptic

degenerate case $e = 0$
circle

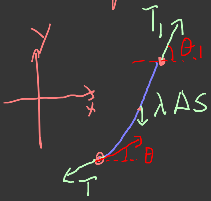
$e = 1 \Rightarrow$ parabolic

$e > 1 \Rightarrow$ hyperbolic

Catenary Problem



λ is the linear density (we multiply by g) so a segment of length Δs, weighs $\Delta W = \lambda \Delta s$



In the x axis, the ballance of forces is:

$$T_1 \cos \theta_1 - T \cos \theta = 0$$

In the y axis, get:

$$T_1 \sin \theta_1 - T \sin \theta - \lambda \Delta s = 0$$

$$\Rightarrow \Delta (T \cos \theta) = 0$$

$$\Delta(T\sin\theta) = \lambda \Delta s$$

$$\Rightarrow \frac{d}{dx}(T\cos\theta) = 0 \quad \Rightarrow T\cos\theta = H$$

$$\frac{d}{dx}(T\sin\theta) = \lambda \frac{ds}{dx}$$

$$\frac{d}{dx}\left(\frac{T\sin\theta}{T\cos\theta}\right) = \frac{\lambda}{H}\frac{ds}{dx} = k\frac{ds}{dx} \qquad k = \frac{\lambda}{H}$$

$$\tan\theta = \frac{dy}{dx} \qquad ds = \sqrt{dx^2 + dy^2}$$

$$\Rightarrow \frac{ds}{dx} = \sqrt{1 + (y')^2}$$

$$\frac{d^2y}{dx^2} = k\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Let $u = y' \Rightarrow$

$$\begin{cases} \dfrac{dy}{dx} = u \\[2ex] \dfrac{du}{dt} = k\sqrt{1+u^2} \end{cases} \Rightarrow k\,dt = \frac{du}{\sqrt{1+u^2}}$$