MAT 303 Lecture 12

Second order motion

$$\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega_0^2 x = \underbrace{f(t)}_{\text{forcing function}}, \quad r \geq 0$$

Cases:

$\alpha]$ $x'' + \omega_0^2 x = 0$ free undamped motion

$\beta]$ $x'' + \omega_0^2 x = f(t)$ forced undamped motion

$\gamma]$ $x'' + 2rx' + \omega_0^2 x = 0$ free damped motion

$\delta]$ $x'' + 2rx' + \omega_0^2 x = f(t)$ forced damped motion

(α) Free undamped motion, aka simple harmonic motion

$$x'' + \omega_0^2 x = 0 \iff \lambda^2 + \omega_0^2 = 0$$

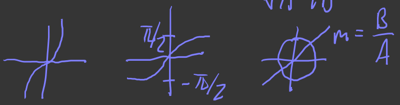
$$\lambda = \pm i\omega_0$$

$$x = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

$$= \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos(\omega_0 t) + \frac{B}{\sqrt{A^2+B^2}}\sin(\omega_0 t)\right)$$

$$= (*)$$

For $\cos(\delta) = \dfrac{A}{\sqrt{A^2+B^2}}$ $\quad \sin(\delta) = \dfrac{B}{\sqrt{A^2+B^2}}$



for $A > 0$, $\delta = \arctan(B/A)$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$e^{i(\alpha+\beta)} = (\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta))$$

$$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$\Rightarrow \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$(*) = \sqrt{A^2+B^2}\cos(\omega_0 t - \delta)$$

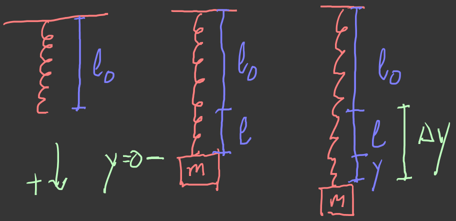
In general the solution is

$$\boxed{x = C\cos(\omega_0 t + \delta)} = C\sin(\omega_0 t + \delta')$$

C is the amplitude, δ is the phase

$T = \frac{2\pi}{\omega_0}$ is the period $\quad \frac{1}{T} = \frac{\omega_0}{2\pi}$ is the frequency

Example:



Newton's 2nd Law:

$$F = ma = m\frac{d^2y}{dt^2}$$

Gravity: mg

$k, m > 0$

Spring force: $-k\,\Delta y$

$$= -k(\ell + y)$$

At equilibrium:

$a = 0 \Rightarrow F = 0$

$\Rightarrow mg - k\ell = 0$

$\Rightarrow mg = k\ell$

In general:

$$m\frac{d^2y}{dt^2} = mg - k(\ell + y)$$

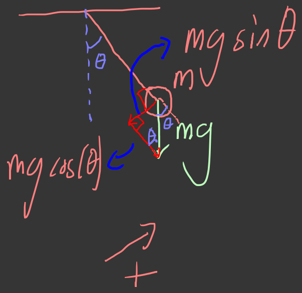
$$= \cancelto{0}{(mg - k\ell)} - ky$$

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

Solutions:

$$y = C\cos\left(\sqrt{\frac{k}{m}}\,t + \delta\right)$$

Pendulum:



$$ma = m\frac{dv}{dt} = -mg\sin(\theta)$$

$$s = l\theta \qquad \frac{ds}{dt} = l\frac{d\theta}{dt}$$

$$\frac{ds}{dt} = v$$

Get equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0 \qquad \Rightarrow \quad ml\frac{d^2\theta}{dt^2} = -mg\sin(\theta)$$

This is a non-linear ODE

When θ is small $\sin(\theta) \simeq \theta$

Small angle approximation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \quad \Rightarrow \quad \theta = A\cos\left(\sqrt{\frac{g}{l}}\,t + \delta\right)$$

So the period is $\simeq 2\pi\sqrt{\frac{l}{g}}$

B] Forced undamped motion

$$x'' + \omega_0^2 x = f(t)$$

assume $f(t) = F\sin(\omega t + \beta)$

$\omega \neq \omega_0$:

ansatz: $x = A\sin(\omega t + \beta)$

$$A(-\omega^2 + \omega_0^2)\sin(\omega t + \beta) = F\sin(\omega t + \beta)$$

$$\Rightarrow A = \frac{F}{\omega_0^2 - \omega^2}$$

$$x = C\sin(\omega_0 t + \delta) + \frac{F}{\omega_0^2 - \omega^2}\sin(\omega t + \beta)$$

If $\frac{\omega_0}{\omega}$ is irrational, then x is not periodic when $C \neq 0$

$\omega = \omega_0$:

$$\lim_{\omega \to \omega_0} \frac{F\left(\sin(\omega t + \beta) - \sin(\omega_0 t + \beta)\right)}{\omega_0^2 - \omega^2}$$

$$= \lim_{\omega \to \omega_0} \frac{F t \cos(\omega t + \beta)}{-2\omega}$$

$$= \frac{-F}{2\omega_0} t \cos(\omega_0 t + \beta)$$

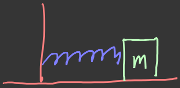
check this satisfies the DE

thus $$x = \underbrace{C \sin(\omega_0 t + \delta)}_{\text{ranges in } [-C, C]} - \underbrace{\frac{F}{2\omega_0} t \cos(\omega_0 t + \beta)}_{\text{as } t \text{ gets big, this term can be big (oscillates between increasingly big numbers)}}$$

This is known as resonance

8] Free damped motion

$$x'' + 2r x' + \omega_0^2 x = 0 \quad r > 0$$



friction

$$F = -\gamma \frac{dx}{dt}$$

$$\lambda^2 + 2r\lambda + \omega_0^2 = 0$$

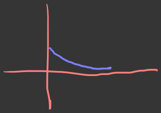
$$\Delta = 4r^2 - 4\omega_0^2$$

$r > \omega_0$ ⇝ distinct real solutions

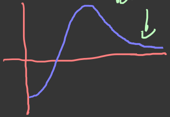
$$\lambda = -r \pm \sqrt{r^2-\omega_0^2} < 0$$

$$x = A e^{\left(-r+\sqrt{r^2-\omega_0^2}\right)t} + B e^{\left(-r-\sqrt{r^2-\omega_0^2}\right)t}$$

these are decaying exponentials:



looks exponential (decaying)



0 or 1 crossing of the x axis, no more!

$r = \omega_0$ $\qquad \lambda = -r$ → still an exponentially decaying term

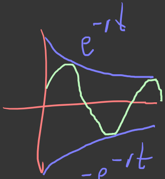
$$x = A e^{-rt} + B t e^{-rt}$$

I think the highest number of x axis crossings possible is 2

$r < \omega_0$ (critically damped)

$$\lambda = -r \pm i\sqrt{\omega_0^2 - r^2}$$

$$x = Ae^{-rt}\cos\left(\sqrt{\omega_0^2-r^2}\,t\right) + Be^{-rt}\sin\left(\sqrt{\omega_0^2-r^2}\,t\right)$$



$$= Ce^{-rt}\cos\left(\sqrt{\omega_0^2-r^2}\,t+\delta\right)$$

crosses the x-axis infinitely often

$T_0 = \dfrac{2\pi}{\omega_0}$ is the undamped (natural) period

$T = \dfrac{2\pi}{\sqrt{\omega_0^2-r^2}}$ is the damped period

this slows down the oscillations

δ] Forced damped motion

$$x'' + 2rx' + \omega_0^2 x = f(t)$$

where $f(t) = F\sin(\omega t+\beta)$

ansatz: $x = A\sin(\omega t+\beta) + B\cos(\omega t+\beta)$

$$-\omega^2 A \sin(\omega t+\beta) + 2r\omega A \cos(\omega t+\beta) + \omega_0^2 A \sin(\omega t+\beta)$$

$$-\omega^2 B \cos(\omega t+\beta) - 2r\omega B \sin(\omega t+\beta) + \omega_0^2 B \sin(\omega t+\beta)$$

$$= (-\omega^2 A - 2r\omega B + \omega_0^2 A) \sin(\omega t+\beta)$$

$$+ (-\omega^2 B + 2r\omega A + \omega_0^2) \cos(\omega t+\beta)$$

$$= F \sin(\omega t+\beta)$$

$$\begin{bmatrix} \omega_0^2-\omega^2 & -2r\omega \\ 2r\omega & \omega_0^2-\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{(\omega_0^2-\omega^2)^2+(2r\omega)^2} \begin{bmatrix} \omega_0^2-\omega^2 & 2r\omega \\ -2r\omega & \omega_0^2-\omega^2 \end{bmatrix} \begin{bmatrix} F \\ 0 \end{bmatrix}$$

$$\Rightarrow A = \frac{F(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+(2r\omega)^2}$$

$$B = \frac{-F(2r\omega)}{(\omega_0^2-\omega^2)^2+(2r\omega)^2}$$

$$\cos(\alpha) = \frac{(\omega_0^2-\omega^2)}{\sqrt{(\omega_0^2-\omega^2)^2+(2r\omega)^2}}$$

$$\sin(\alpha) = \frac{(2r\omega)}{\sqrt{(\omega_0^2-\omega^2)^2+(2r\omega)^2}}$$

$$X = \frac{F}{\sqrt{(\omega_0^2-\omega^2)^2+(2r\omega)^2}} \times \left(\cos(\alpha)\sin(\omega t+\beta) - \sin(\alpha)\cos(\omega t+\beta)\right)$$

$$= \boxed{\frac{F}{\sqrt{(\omega_0^2-\omega^2)^2+(2r\omega)^2}} \sin(\omega t+\beta-\alpha)}$$

steady state motion

The solutions to the homogeneous eqn

?

depend on $r \gtreqless \omega_0$

independent of the case, the homogenous part decays exponentialy

The steady state solution has period $\frac{2\pi}{\omega}$

the amplitude is $\frac{F}{\sqrt{(\omega_0^2-\omega^2)^2+(2r\omega)^2}}$

Consider $g(\omega) = (\omega_0^2-\omega^2)^2+(2r\omega)^2$

$$g'(\omega) = 4\omega(\omega^2-\omega_0^2)+8r^2\omega$$

$g'(\omega) = 0$ when

$$\omega_0^2-\omega^2=2r^2$$

$$\Rightarrow$$

$$0 = \omega^2-\omega_0^2+2r^2$$

$\omega = \pm\sqrt{\omega_0^2-2r^2}$ assuming $\omega_0^2 \geq 2r^2$

damped resonant frequency

F

$$A_{max} = \frac{\quad}{\sqrt{4r^4 + 4r^2(w_0^2 - 2r^2)}}$$

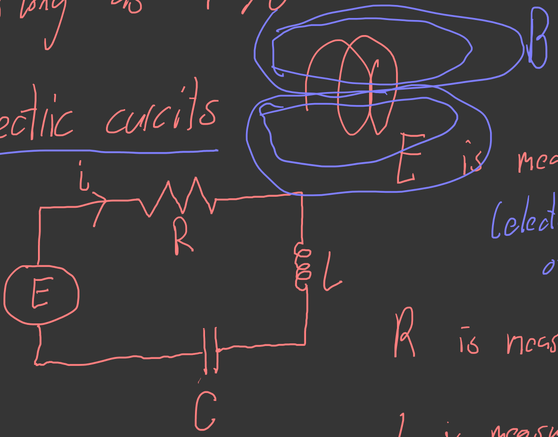
$$= \frac{F}{\sqrt{4r^2w_0^2 - 4r^4}}$$

$$= \frac{F}{2r\sqrt{w_0^2 - r^2}} \qquad w_0 \geq r$$

As $r \to 0$, $A_{max} \to \infty$

So long as $r > 0$ A_{max} is finite

Electric circuits



B stores energy

E is measured in volts (electromotive force of the circuit)

R is measured in ohms

L is measured in henrys

E as an energy source and the rest of the things as energy sinks measured by the voltage drop across the component

C is measured in farads

i is measured in ampres

R resists current

L resist changes in current

C makes it harder to accumulate a charge as the charge gets bigger

Formulas

$$\Delta V_{res} = Ri$$

$$\Delta V_{ind} = L\frac{di}{dt}$$

$$\Delta V_{cap} = \frac{1}{C}q$$

where $\frac{dq}{dt} = i$

$$\begin{cases} V = Ri \\ L\frac{di}{dt} = V \\ C\frac{dv}{dt} = i \end{cases}$$

dual

Kirchhoff's Voltage Law:

The voltage changes in a loop sum to zero

$$E = Ri + L\frac{di}{dt} + \frac{1}{C}q$$

$$\Rightarrow L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dE}{dt}$$

This is a second order linear ODE

if $E(t) = F\omega\cos(\omega t + \beta)$

then (assuming complex roots)

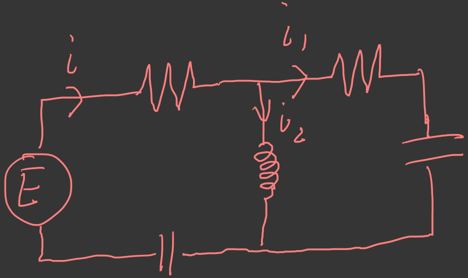
$$i = Ce^{-\frac{R}{2L}t}\sin\left(\frac{\sqrt{4CL - R^2C^2}}{2CL}t + \delta\right)$$

$$+ \frac{F\omega C}{\sqrt{(R\omega C)^2 + (1 - CL\omega^2)^2}}\sin(\omega t + \beta + \alpha)$$

$$Z = \sqrt{R^2 + \left(\frac{1}{\omega C} - L\omega\right)^2}$$

is the impedence of the cucit and the steady state amplitude is inversely proportional to this

$$A = \frac{F}{Z}$$



Next time: