MAT 303 Lecture 11

Table of Laplace Transforms

f	$\mathcal{L}[f]$
1	$1/s$
$t^n \quad n \in \mathbb{N}$	$n!/s^{n+1}$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$f^{(n)}$	$s^n \mathcal{L}[f] - s^{n-1} f(0)$ $- s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$
$t^n f$	$(-1)^n \frac{d^n}{ds^n} \mathcal{L}[f]$
$e^{at} f$	$\mathcal{L}[f](s-a)$

Last time:

Example: Find $\mathcal{L}[y]$ for y is a solution of

$$y'' + 2y' + 5y = e^{-t}, \quad y(0) = 1, \; y'(0) = -3$$

Sol'n: $\mathcal{L}[y] = \dfrac{s^2}{(s+1)(s^2+2s+5)}$

Now we can find y:

$$\frac{s^2}{(s+1)\underbrace{(s^2+2s+5)}_{(s+1)^2+4}} = \frac{A}{s+1} + \frac{B(s+1)+C}{(s+1)^2+4} = (*)$$

$$s^2 = A(s^2+2s+5) + B(s^2+2s+1) + C(s+1)$$

$$= (A+B)s^2 + (2A+2B+C)s + (5A+B+C)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \\ 5 & 1 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & -4 & 1 & -5 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -4 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 3/4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$(*) = \frac{1}{4}\,\frac{1}{s+1} + \frac{3}{4}\,\frac{s+1}{(s+1)^2+4} - \frac{2}{(s+1)^2+4}$$

$$y = \mathcal{L}^{-1}\left[\mathcal{L}[y]\right] \qquad \boxed{\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+4}\right] = e^{-t}\,\mathcal{L}\left[\frac{s}{s^2+4}\right]}$$

$$= \frac{1}{4}e^{-t} + \frac{3}{4}e^{-t}\cos(2t) - e^{-t}\sin(2t)$$

Example 2: $y^{(4)} - y = 0$

$y(0) = 0 \quad y''(0) = 0$

$y'(0) = 0 \quad y'''(0) = 1$

$$\mathcal{L}\left[y^{(4)} - y\right] = s^4\,\mathcal{L}[y] - 1 - \mathcal{L}[y]$$

$$\mathcal{L}[0] = 0$$

$$\Rightarrow (s^4-1)\,\mathcal{L}[y] = 1 \quad \Rightarrow \quad \mathcal{L}[y] = \frac{1}{s^4-1}$$

$$\frac{1}{s^4-1} = \frac{1}{(s^2-1)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1} = (*)$$

$$\Rightarrow 1 = A\,\overbrace{(s+1)(s^2+1)}^{s^3+s^2+s+1} + B\,\overbrace{(s-1)(s^2+1)}^{s^3-s^2+s-1} + \overbrace{(Cs+D)(s^2-1)}^{Cs^3+Ds^2-Cs-D}$$

$$= (A+B+C)s^3 + (A-B+D)s^2 + (A+B-C)s + (A-B-D)$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & -2 & -1 & -1 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 0 & -1/4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/2 \end{array}\right]$$

$$(*) = \frac{1}{4}\frac{1}{s-1} - \frac{1}{4}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s^2+1}$$

$$y = \mathcal{L}^{-1}[\mathcal{L}[y]] = \frac{1}{4}e^{t} - \frac{1}{4}e^{-t} - \frac{1}{2}\sin(t)$$

Laplace transforms for systems

$$x' = ax + by + f(t)$$

$$y' = cx + dy + g(t)$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \vec{f} = \begin{bmatrix} f \\ g \end{bmatrix}$$

$$x(0) = x_0$$

$$y(0) = y_0$$

$$\vec{x}' = A\vec{x} + \vec{f}$$

$$\vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$s\mathcal{L}[x] - x_0 = a\mathcal{L}[x] + b\mathcal{L}[y] + \mathcal{L}[f]$$

$$s\mathcal{L}[y] - y_0 = c\mathcal{L}[x] + d\mathcal{L}[y] + \mathcal{L}[g]$$

$$(s-a)\mathcal{L}[x] - b\mathcal{L}[y] = \mathcal{L}[f] + x_0$$

$$-c\mathcal{L}[x] + (s-d)\mathcal{L}[y] = \mathcal{L}[g] + y_0$$

$$(sI - A)\mathcal{L}[\vec{x}] = \mathcal{L}[\vec{f}] + \vec{x}_0$$

$$\mathcal{L}[\vec{x}] = (sI - A)^{-1}\left(\mathcal{L}[\vec{f}] + \vec{x}_0\right)$$

Example

$$x' = -3x + 4y + \sin(t) \qquad x(0) = 0$$

$$y' = -2x + 3y + t \qquad y(0) = 1$$

$$\begin{bmatrix} s+3 & -4 \\ 2 & s-3 \end{bmatrix} \begin{bmatrix} \mathcal{L}[x] \\ \mathcal{L}[y] \end{bmatrix} = \begin{bmatrix} \frac{1}{s^2+1} \\ \frac{1}{s^2} + 1 \end{bmatrix}$$

$$\begin{bmatrix} \mathcal{L}[x] \\ \mathcal{L}[y] \end{bmatrix} = \frac{1}{s^2-9+8} \begin{bmatrix} s-3 & 4 \\ -2 & s+3 \end{bmatrix} \begin{bmatrix} \frac{1}{s^2+1} \\ \frac{s^2+1}{s^2} \end{bmatrix}$$

$$= \frac{1}{s^2-1} \begin{bmatrix} \frac{s-3}{s^2+1} + \frac{4s^2+4}{s^2} \\ \frac{-2}{s^2+1} + \frac{s^3+3s^2+s+3}{s^2} \end{bmatrix}$$

$$s^5 + 3s^4 + s^3 + 3s^2$$
$$+ s^3 + 3s^2 + s + 3$$

$$= \frac{1}{(s^2-1)(s^2+1)s^2} \begin{bmatrix} s^3 - 3s^2 + 4s^4 + 8s^2 + 4 \\ -2s^2 + s^5 + 3s^4 + 2s^3 + 6s^2 + s + 3 \end{bmatrix}$$

$$= \frac{1}{(s^2-1)(s^2+1)s^2} \begin{bmatrix} 4s^4+s^3+5s^2+4 \\ s^5+3s^4+2s^3+4s^2+s+3 \end{bmatrix}$$

Then solve two 6th order systems to get the partial fractions decompositions...

New stuff:

its a second order linear inhomogeneous system

$$x'' + y' + 2x = 0$$

$$2x' - y' = \cos(t)$$

$$x(0) = 0 \quad x'(0) = 0 \quad y(0) = 0$$

$$s^2 \mathcal{L}[x] + s\mathcal{L}[y] + 2\mathcal{L}[x] = 0$$

$$2s\mathcal{L}[x] - s\mathcal{L}[y] = \frac{s}{s^2+1}$$

$$(s^2+2)\mathcal{L}[x] + s\mathcal{L}[y] = 0$$

$$2s\,\mathcal{L}[x] - s\,\mathcal{L}[y] = \frac{s}{s^2+1}$$

$$\begin{bmatrix} (s^2+2) & s \\ 2s & -s \end{bmatrix} \begin{bmatrix} \mathcal{L}[x] \\ \mathcal{L}[y] \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{s}{s^2+1} \end{bmatrix}$$

$$\begin{bmatrix} \mathcal{L}[x] \\ \mathcal{L}[y] \end{bmatrix} = \frac{1}{-s^3-2s-2s^2} \begin{bmatrix} -s & -s \\ -2s & (s^2+2) \end{bmatrix} \begin{bmatrix} 0 \\ \frac{s}{s^2+1} \end{bmatrix}$$

$$= \frac{-1}{s(s^2+2s+2)} \begin{bmatrix} \frac{-s^2}{s^2+1} \\ \frac{s^3+2s}{s^2+1} \end{bmatrix} = \begin{bmatrix} \frac{s}{((s+1)^2+1)(s^2+1)} \\ \frac{-s^2-2}{((s+1)^2+1)(s^2+1)} \end{bmatrix}$$

$$\mathcal{L}[x] = \frac{s}{((s+1)^2+1)(s^2+1)} = \frac{A(s+1)+B}{(s+1)^2+1} + \frac{Cs+D}{s^2+1} = (*)$$

$$s = A(s^3+s^2+s+1) + B(s^2+1) + C(s^3+2s^2+2s)$$
$$+ D(s^2+2s+2)$$

$$= (A+C)s^3 + (A+B+2C+D)s^2$$

$$+ (A + 2C + 2D)s + (A + B + 2D)$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 \\ 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 0 & 2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & -1 & 2 & 0 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2/5 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1/5 \\ 0 & 1 & 0 & 0 & -3/5 \\ 0 & 0 & 1 & 0 & 1/5 \\ 0 & 0 & 0 & 1 & 2/5 \end{array}\right]$$

$$(*) = -\frac{1}{5}\,\frac{s+1}{(s+1)^2+1} - \frac{3}{5}\,\frac{1}{(s+1)^2+1}$$

$$+ \frac{1}{5}\,\frac{s}{s^2+1} + \frac{2}{5}\,\frac{1}{s^2+1}$$

$$\Rightarrow x = -\frac{1}{5}e^{-t}\cos(t) - \frac{3}{5}e^{-t}\sin(t) + \frac{1}{5}\cos(t) + \frac{2}{5}\sin(t)$$

Recall

$$2\mathcal{L}[x] - \mathcal{L}[y] = \frac{1}{s^2+1}$$

$$\Rightarrow \mathcal{L}[y] = 2\mathcal{L}[x] + \frac{1}{s^2+1}$$

$$= -\frac{2}{5}\frac{s+1}{(s+1)^2+1} - \frac{6}{5}\frac{1}{(s+1)^2+1} + \frac{2}{5}\frac{s}{s^2+1} + \frac{9}{5}\frac{1}{s^2+1}$$

$$\Rightarrow y = -\frac{2}{5}e^{-t}\cos(t) - \frac{6}{5}e^{-t}\sin(t) + \frac{2}{5}\cos(t) + \frac{9}{5}\sin(t)$$

Define $(f*g)(t) = \int_0^t f(t-\xi)\,g(\xi)\,d\xi$

this is the convolution of f and g

Theorem: $\mathcal{F}[f*g] = \mathcal{F}[f]\,\mathcal{F}[g]$

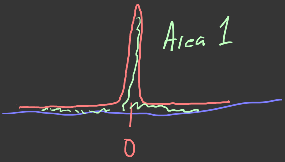
Cor: $\mathcal{F}^{-1}[FG] = \mathcal{F}^{-1}[F] * \mathcal{F}^{-1}[G]$

$f*g = g*f$ $\qquad$ $f*(g*h) = (f*g)*h$

bilinear $\qquad$ (unit is not a function)

$,\infty$

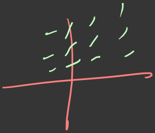
unit: $\qquad$ distribution $\Big\}\ \delta_0$



Review:

First order systems

$$y' = f(x,y)$$



At every (a,b) get a slope $f(a,b)$

Solutions are integral curves of the distribution

Seperable: $y' = \frac{f(x)}{g(y)} \Rightarrow g(y)\,dy = f(x)\,dx$

$$\Rightarrow \int g(y)\,dy = \int f(x)\,dx + C$$

To get y as a function of x, often take c to be $\log(C)$ or similar

if $\frac{dy}{y} = f(x)\,dx \Rightarrow \log(y) = \int f(x)\,dx + \log(C)$

$$\Rightarrow y = C e^{\int f(x)\,dx}$$

Linear Equations: $y' + f(x)\,y = g(x)$

Homogeneous case: $y' = -f(x)y$ is seperable ✓

Inhomogeneous case: use variation of parameters

let y_1 be a solution $y' + f(x)y = 0$

ansatz: $y = u y_1$

$$u' y_1 + \cancelto{0}{\underbrace{u y_1' + u f(x) y_1}} = g(x)$$

$$\Rightarrow u' = \frac{g}{y_1} \quad \Rightarrow u = \int \frac{g(x)}{y_1(x)} \, dx$$

On the exam start with the ansatz, not this integral.

Bernoulli equations: $y' + f(x) y = g(x) y^{\alpha}$

then you just solve the homogeneous eqn and use the same ansatz $y = u y_1$

Homogeneous Equations: $y' = \varphi\left(\frac{y}{x}\right)$

use the ansatz $u = \frac{x}{y}$ or $u = \frac{y}{x}$

if using $u = \frac{y}{x} \Leftrightarrow y = ux$

$$u' x + u = \varphi(u)$$

$$u' = \frac{\varphi(u) - u}{x} \Rightarrow \frac{du}{\varphi(u) - u} = \frac{dx}{x}$$ separable ✓

Exact equations:

$$y' = f(x,y)$$

$$\Leftrightarrow dy - f(x,y)dx = 0$$

$$\Leftrightarrow \mu dy - \mu f(x,y)dx = 0$$

We will consider $M(x,y)dx + N(x,y)dy = 0$

Recall $dF = F_x dx + F_y dy$

where $F_x = \frac{\partial F}{\partial x}$ $F_y = \frac{\partial F}{\partial y}$

The equation $dF = 0$ has solutions $F(x,y) = C$

When is $Mdx + Ndy$ exact (assuming defined on rectangle)

A! $M_y = N_x$

Suppose this holds then want F with

$F_x = M$ $F_y = N$

Observation: for $V(x,y) = \int_{x_0}^{x} M(s,y)ds$

$$\frac{\partial(F-V)}{\partial x} = 0 \Rightarrow F = \int_{x_0}^{x} M(s,y)\,ds + \phi(y)$$

Then $F_y = \frac{\partial}{\partial y} V + \phi'(y)$

This lets you solve for $\phi'(y)$, then integrate, then plug ϕ back in to get F

Integrating factors: If $M\,dx + N\,dy = 0$ is not exact, we will try to make

$$\mu M\,dx + \mu N\,dy = 0 \text{ exact}$$

Cases:

$$\left.\begin{aligned} \frac{N_x - M_y}{-N} &= f(x) \\[1em] \frac{N_x - M_y}{M} &= f(y) \\[1em] \frac{N_x - M_y}{xM - yN} &= f(xy) \end{aligned}\right\} \text{solve integral gives } \mu \text{ to make the eq'n exact}$$

Second order theory:

Linear second order ODE: $y'' + f y' + g y = h$

Homogeneous case, with constant coefficients:

for a n-th order eqn

$$a_n y^{(n)} + \dots + a_1 y' + a_0 y = 0$$

ansatz: $y = e^{\lambda x}$

$$(a_n \lambda^n + \dots + a_0) e^{\lambda x} = 0$$

If λ is complex make real solutions (see notes)

$\Rightarrow y = e^{\lambda x}$ is a solution exactly when λ is a root of the charicteristic eq'n

Repeated roots: m-th order root λ

$$e^{\lambda x}, x e^{\lambda x}, \dots, x^{m-1} e^{\lambda x}$$

Wronskian $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

We saw that if y_1 is a non-zero solution of

$$y'' + f y' + g y = 0$$

Then we can use W to find y_2 such that the general solution is $C y_1 + C y_2$

Remember that linearity is a thing, so for homogeneous linear eq'ns get a vector space of solutions. (affine space in inhomogeneous case)

Euler equation: $a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_0 y = 0$

A substitution works here (see notes)

Inhomogeneous linear equations:

Variation of parameters

For second order systems:

Let y_1, y_2 be lin indep solutions of

$$y'' + f y' + g y = 0$$

Then use ansatz $y = u_1 y_1 + u_2 y_2$
when solving $y'' + f y' + g y = h$

add the equation $\left\{ u_1' y_1 + u_2' y_2 = 0 \right.$
get equation $\left. u_1' y_1' + u_2' y_2' = h \right.$

this is a system for u_1', u_2'

solve, get u_1, u_2 by integrating, plug in

For third order systems: see L9 notes

Undetermined Coefficients: If the inhomogeneous part is $ae^{\lambda x}$ or $a\cos(bx)$, then there is a shortcut method, instead of variation of parameters (see notes)

Linear systems:

Constant coefficient first order systems

$$\vec{x}' = A\vec{x} + \vec{f}$$

Homogeneous case:

ansatz $\vec{x} = \vec{v}\, e^{\lambda t}$

we got that this is a solution iff $\vec{v}$ is an eigenvector of A with eigenvalue λ

First I find the eigenvalues by solving
$$\det(A - \lambda I) = 0 \quad \leftrightsquigarrow \quad n\text{th order poly in } \lambda$$

For every distinct eigenvalue, get at least one eigenvector

For repeated roots, you in general get:
$$\dots \vec{v}_2 \xmapsto{A-\lambda I} \vec{v}_1 \xmapsto{A-\lambda I} \vec{0}$$

in this case we have a special ansatz (see notes)

Summary: Get n lin indep solutions

Inhomogeneous systems:

Given lin indep solutions $\vec{y}_1, \dots, \vec{y}_n$ of the homogeneous system

Construct $X = [\vec{y}_1 \cdots \vec{y}_n]$

then do variation of parameters (have to invert X along the way)

(see notes)

End of examinable midterm material