MAT 303 Lecture 10

Laplace transforms

(another way to solve constant coefficient linear eqn's with inhomogeneous part $t^n e^{ax}$ or with cos/sin, etc)

Suppose f is defined for $t \geq 0$

$$\mathcal{L}[f](s) = \int_0^\infty e^{-ts} f(t)\,dt$$

First observation: $\mathcal{L}$ is linear (hence an operator)

$$\mathcal{L}[f+\lambda g](s) = \int_0^\infty e^{-ts}(f+\lambda g)\,dt$$

$$= \int_0^\infty e^{-ts} f(t)\,dt + \lambda \int_0^\infty e^{-ts} g(t)\,dt$$

$$= \mathcal{L}[f] + \lambda \mathcal{L}[g]$$

Second observation: If $\mathcal{L}[f](a_0)$ converges then $\mathcal{L}[f](s)$ converges for $s > a_0$.

(Assuming (piecewise) continuity)

This integral does not converge when there is infinite area under the curve, i.e. when $\lim\limits_{x\to\infty} \int_0^x e^{-st}|f(t)|dx$

$= \infty$

But by changing e^{-ta_0} to e^{-ts} for $s > a_0$ we divide by $e^{(s-a_0)t}$. On $t \geq 0$, this is >1, so this helps with convergence.

When does $\mathcal{L}[f]$ converge at all?

A! If f has exponential order, i.e.

$$\lim_{t\to\infty} \frac{f(t)}{e^{at}} = 0 \quad \text{for some value of } a$$

e.g. e^{at} is of exponential order for any a

as $\lim_{t\to\infty} \frac{e^{at}}{e^{bt}} = 0$ if $b > a$

counterexample: $\begin{cases} t^t = e^{\log(t)t} \\ e^{t^2} \end{cases}$ $\quad \frac{e^{\log(t)t}}{e^{at}} = e^{\overbrace{(\log(t)-a)}^{\text{this gets big}}t}$

$\{ e^{e^x}$ even worse!!!

Examples:

$$\mathcal{L}[1](s) = \int_0^\infty e^{-ts}\,dt = \frac{-1}{s} e^{-ts} \Big|_0^\infty$$

$$= 0 - \left(\frac{-1}{s}\right) = \frac{1}{s} \quad \text{for } s > 0$$

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-ts} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \frac{-1}{s-a} e^{-(s-a)t}\Big|_0^\infty = \frac{1}{s-a} \quad \text{for } s>a$$

$$\mathcal{L}\left[e^{(\alpha+i\beta)t}\right] = \int_0^\infty e^{-ts} e^{(\alpha+i\beta)t}\,dt$$

$$= \int_0^\infty e^{-(s-\alpha-i\beta)t} = \frac{-1}{s-\alpha-i\beta} e^{-(s-\alpha-i\beta)t}\Big|_0^\infty$$

$$= (*)$$

$e^{-i\beta t} = \cos(\beta t) - i\sin(\beta)$ is on the unit circle in the complex plane

mult by $e^{-(s-\alpha)t}$ in the limit $t \to \infty$

want $s-\alpha > 0$ in order for $\lim_{t\to\infty}$ to converge

$(*) = \frac{1}{s-(\alpha+i\beta)}$ for $s > \alpha$

$[c(\beta t) + is(\beta t)]$
$-(c(\beta t) - is(\beta t))$

$$\mathcal{L}[\sin(\beta t)] = \mathcal{L}\left[\frac{e^{i\beta t} - e^{-i\beta t}}{2i}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s-i\beta} - \frac{1}{s+i\beta}\right]$$

$$= \frac{1}{2i} \frac{s+i\beta - (s-i\beta)}{s^2+\beta^2} = \frac{\beta}{s^2+\beta^2} \quad s>0$$

$$\mathcal{L}[\cos(\beta t)] = \mathcal{L}\left[\frac{e^{i\beta t} + e^{-i\beta t}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-i\beta} + \frac{1}{s+i\beta}\right]$$

$$= \frac{1}{2} \frac{s+i\beta + s-i\beta}{s^2+\beta^2} = \frac{s}{s^2+\beta^2}$$

Let $f(t) = 2 + 5e^{-2t} - 3\sin(4t)$

$$\mathcal{L}[f] = 2\mathcal{L}[1] + 5\mathcal{L}[e^{-2t}] - 3\mathcal{L}[\sin(4t)]$$

$$= \frac{2}{s} + \frac{5}{s+2} - \frac{12}{s^2+16}$$

Third Observation: Let's take $\frac{d^n}{ds^n}\mathcal{L}[f]$

$$\frac{d^n}{ds^n}\mathcal{L}[f] = \frac{d^n}{ds^n}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{\partial^n}{\partial s^n} e^{-st} f(t)\,dt$$

$$= \int_0^\infty (-t)^n e^{-st} f(t)\,dt = (-1)^n \int_0^\infty e^{-st} t^n f(t)\,dt$$

$$= (-1)^n \mathcal{L}[t^n f]$$

$$\Rightarrow \boxed{\mathcal{L}[t^n f] = (-1)^n \frac{d^n}{ds^n}\mathcal{L}[f]} \quad n \in \mathbb{N}$$

Cor: $\mathcal{L}[t^n] = (-1)^n \frac{d^n}{ds^n}\mathcal{L}[1] = (-1)^n \left(\frac{1}{s}\right)^{(n)}$

$= (*)$

$$\left(\frac{1}{s}\right)^{(0)} = \frac{1}{s} \qquad \left(\frac{1}{s}\right)' = \frac{-1}{s^2} \qquad \left(\frac{1}{s}\right)'' = \frac{(-1)(-2)}{s^3}$$

$$\left(\frac{1}{s}\right)''' = \frac{(-1)(-2)(-3)}{s^4} \qquad \left(\frac{1}{s}\right)^{(n)} = \frac{(-1)^n n!}{s^{n+1}}$$

$$(*) = \frac{n!}{s^{n+1}} \qquad (\text{for } n \in \mathbb{N} = \{0, 1, 2, \dots\})$$

Fourth Observation:

$$\mathcal{L}\left[e^{at} f\right](s) = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt = \mathcal{L}[f](s-a)$$

(is reminiscent of the exponential shift theorem!)

Examples:

$$\mathcal{L}\left[e^{-2t} \sin(4t)\right](s) = \mathcal{L}\left[\sin(4t)\right](s+2)$$

$$= \frac{4}{\underbrace{(s+2)}_{}^2 + \underbrace{16}_{}} = \frac{4}{s^2 + 4s + 20}$$

$$\Delta = b^2 - 4ac$$

sum of squares

Aside: If we have $\frac{f}{ax^2+bx+c}$

if $\Delta < 0$
then we
complete the square

if $\Delta \geq 0$
we can factor
via quadratic eq'n

$$a\left(x^2 + b/a\, x + c/a\right)$$

$$(x+\alpha)^2 = x^2 + 2\alpha x + \alpha^2$$

we want $b/a = 2\alpha \Rightarrow \alpha = \frac{b}{2a}$

$$a\left(x^2 + b/a\, x + c/a\right) = a\left(\left(x + \frac{b}{2a}\right)^2 + \underbrace{\frac{c}{a} - \frac{b^2}{4a^2}}_{\frac{-\Delta}{4a^2}}\right)$$

if $\Delta < 0$ then we get
a sum of squares

$\frac{-\Delta}{4a^2}$
is > 0 iff
$\Delta < 0$

$$\mathcal{L}\left[t^2 \cos(3t)\right] = \frac{d^2}{ds^2} \mathcal{L}\left[\cos(3t)\right]$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+9}\right) = \frac{d}{ds}\left(\frac{\overbrace{s^2+9-2s^2}^{9-s^2}}{(s^2+9)^2}\right)$$

$$= \frac{-2s(s^2+9)^2 - 4s(s^2+9)(9-s^2)}{(s^2+9)^4}$$

$$= \frac{2s\left(-s^2-9+2s^2-18\right)}{(s^2+9)^3} = \frac{2s\left(s^2-27\right)}{(s^2+9)^3}$$

for s real:

$$\mathcal{L}[f] = \int_0^\infty e^{-ts} f(t)\,dt$$

$$\mathrm{Re}\left(\mathcal{L}[f]\right) = \int_0^\infty e^{-ts}\,\mathrm{Re}(f(t))\,dt$$

$$= \mathcal{L}[\mathrm{Re}(f)]$$

second approach:

$$\cos(3t) = \mathrm{Re}\left(e^{i3t}\right)$$

$$\mathcal{L}\left[t^2\cos(3t)\right] = \mathcal{L}\left[\mathrm{Re}\left(t^2 e^{i3t}\right)\right]$$

$$= \mathrm{Re}\left(\mathcal{L}\left[t^2 e^{i3t}\right](s)\right)$$

$$= \mathrm{Re}\left(\mathcal{L}\left[t^2\right](s-3i)\right)$$

$$= \mathrm{Re}\left(\frac{2}{(s-3i)^3}\right) = \mathrm{Re}\left(\frac{2(s+3i)^3}{(s^2+9)^3}\right) = (*)$$

$$(s+3i)^3 = \underline{\underline{s}}^2 + 3s^2(3i) + \underline{\underline{3s\,(3i}})^2 + (3i)^3$$

$$\binom{3}{0}=1 \quad \binom{3}{1}=3 \quad \binom{3}{2}=3 \quad \binom{3}{3}=1$$

1

1 1

1 2 1

1 3 3 1

$$(*) = \frac{2(s^2-27s)}{(s^2+9)^3} = \frac{2s(s^2-27)}{(s^2+9)^3}$$

The second approach is clearly better, particularly when we have $t^n\cos(\beta x)$ for $n \geq 3$

Fifth observation: suppose both $\mathcal{L}[f]$ and $\mathcal{L}[f']$ converge at s

$$\mathcal{L}[f'] = \int_0^\infty e^{-ts} f'(t)\,dt \quad \begin{bmatrix} u = e^{-ts} & dv = f'(t)dt \\ du = -se^{-ts}dt & v = f(t) \end{bmatrix}$$

$$= e^{-ts}f(t)\Big|_0^\infty + s\int_0^\infty e^{-ts}f(t)\,dt$$

$$= -f(0) + s\,\mathcal{L}[f] = s\,\mathcal{L}[f] - f(0)$$

Cor: $\mathcal{L}[f''] = s\,\mathcal{L}[f'] - f'(0)$

$$= s^2\mathcal{L}[f] - s\,f(0) - f'(0)$$

$$\mathcal{L}[f^{(n)}] = s^n\mathcal{L}[f] - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

Initial Value Problems:

If you have a n-th order ODE, $y^{(n)} = F(t, y, .., y^{(n-1)})$ the general solution will have n parameters.

If we want a unique solution, we should give n equations. (moral idea)

idea 1: $y(0) = A \quad y(1) = B \quad ... \quad y(n-1) = \alpha_n$

this is not a good set of n equations (might have multiple solutions, even a space of sol'n of dim > 0)

idea 2: $y(0) = \alpha_1 \quad y'(1) = \alpha_2 \quad .... \quad y^{(n-1)}(n-1) = \alpha_n$

better, but still not perfect

correct idea: $y(0) = \alpha_1 \quad y'(0) = \alpha_2 \quad ... \quad y^{(n-1)}(0) = \alpha_n$

then $y^{(n)} = F(t, y, .., y^{(n-1)})$ determines $y^{(n)}(0)$ and also by differentiating the above, get all higher derivatives (follows from multivariable chain rule)

so we know all of the taylor coefficients

intuitively this determines the funtion
assuming little more than continuity of F, get uniquness
(MATH THEOREM)

Nature evolves in an unique way!
Uniquness is obvious to physicists.

Example: Find $\mathcal{L}[y]$ for y is a solution of

$$y''+2y'+5y=e^{-t}, \quad y(0)=1, \ y'(0)=-3$$

Soln: $\mathcal{L}[y''+2y'+5y]$

$$= s^2\mathcal{L}[y]-s\overbrace{y(0)}^{1}-\overbrace{y'(0)}^{-3}+2(s\mathcal{L}[y]-\overbrace{y(0)}^{1})+5\mathcal{L}[y]$$

$$=(s^2+2s+5)\mathcal{L}[y]-s+1$$

$$\mathcal{L}[e^{-t}]=\frac{1}{s+1}$$

$$\Rightarrow (s^2+2s+5)\mathcal{L}[y]=\frac{1}{s+1}+s-1$$

$$= \frac{1+s^2+s-(s+1)}{s+1} = \frac{s^2}{s+1}$$

$$\Rightarrow \mathcal{L}[y] = \frac{s^2}{(s+1)(s^2+2s+5)}$$

Inverse Laplace Transforms

$\mathcal{L}$ is injective, meaning that if $\mathcal{L}[f] = \mathcal{L}[g]$ then $f=g$ (on $t \geq 0$)

Thus if we know $\mathcal{L}[y] = F$ and we find g with $\mathcal{L}[g] = F \Rightarrow g = y$

So any "Laplace antiderivative" is the unique inverse, and so we get a sol'n to the diff eqn satisfying the inital conditions!

Notation: $\mathcal{L}^{-1}[F]$

Examples:

(i) $\mathcal{L}^{-1}\left[\frac{4}{s^2+16}\right] = \sin(4t)$

(ii) $\mathcal{L}^{-1}\left[\frac{6}{(s+2)^4}\right] = e^{-2t}\mathcal{L}^{-1}\left[\frac{6}{s^4}\right] = t^3e^{-2t}$

$\mathcal{L}[t^3] = \frac{6}{s^4}$ $\quad$ $\mathcal{L}\left[e^{at}\mathcal{L}^{-1}[f]\right] = \mathcal{L}\left[\mathcal{L}^{-1}[f]\right](s-a)$

$$= f(s-a)$$

$$\Rightarrow e^{at}\mathcal{L}^{-1}[f] = \mathcal{L}^{-1}[f(s-a)]$$

Linearity:

$$\left.\begin{aligned} &\mathcal{L}^{-1}[F] = f \quad \mathcal{L}^{-1}[G] = g \\ &\Rightarrow \mathcal{L}[f] = F \quad \mathcal{L}[g] = G \\ &\mathcal{L}[f+\lambda g] = F+\lambda G \\ &\Rightarrow \mathcal{L}^{-1}[F+\lambda G] = f+\lambda g \end{aligned}\right\}$$

This shows $\mathcal{L}^{-1}$ is linear!

(iii) $\mathcal{L}^{-1}\left[\frac{s+1}{s^2+2s+5}\right] = \mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+4}\right]$

$$= e^{-t}\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = e^{-t}\cos(2t)$$

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