MAT 303: Calculus IV

Instructor: Astra Kolomatskaia (she/her)

astra.kolomatskaia@gmail.com ← Preferred contact address

Integration Review (Part I)

Chain rule: $(f\circ g)'(x) = f'(g(x))g'(x)$

↳ denotes composition

$(f\circ g)(x) = f(g(x))$

Hence: $f(g(x)) = \int (f\circ g)'(x)\,dx = \int f'(g(x))g'(x)\,dx$

Suppose $F' = f$

then $\int f(g(x))g'(x)\,dx = F(g(x)) + C$

Denoted by: (substitution)

$$\int f(g(x))g'(x)\,dx \left[\begin{array}{l}\text{Let } u = g(x)\\ du = g'(x)\,dx\end{array}\right] = \int f(u)\,du = F(u) + C$$

$$= F(g(x)) + C \quad \square$$

Examples:

$$\int \frac{2x}{x^2+1}\,dx \left[\begin{array}{l}\text{Let } u = x^2+1\\ du = 2x\,dx\end{array}\right] = \int \frac{du}{u} = \log|u| + C$$

$$= \log|x^2+1| + C \quad \square$$

$$\int \frac{x}{x^4+1}\,dx \quad \begin{bmatrix} \text{Let } u = x^2 \\ du = 2x\,dx \end{bmatrix} = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\arctan(u) + C$$

$$= \frac{1}{2}\arctan(x^2) + C \quad \square$$

$$\int \sqrt{1+e^x}\,dx \quad \begin{bmatrix} \text{Let } u = 1+e^x \\ du = e^x\,dx \end{bmatrix} = \int \frac{\sqrt{1+e^x}}{e^x}\,e^x\,dx$$

$$= \int \frac{\sqrt{1+e^x}}{(1+e^x)-1}\,e^x\,dx = \int \frac{\sqrt{u}}{u-1}\,du \quad \overset{u = w^2}{\begin{bmatrix} w = \sqrt{u} \\ dw = \frac{du}{2\sqrt{u}} \end{bmatrix}} = \int \frac{2u}{u-1}\,\frac{1}{2\sqrt{u}}\,du$$

$$= \int \frac{2w^2}{w^2-1}\,dw = \int \frac{2(w^2-1)+2}{w^2-1}\,dw = \int \left[2 + \frac{2}{w^2-1}\right] dw = (*)$$

$$\boxed{\begin{aligned}
& w^2 - 1 = (w-1)(w+1) \\
& \frac{A}{w-1} + \frac{B}{w+1} = \frac{A(w+1) + B(w-1)}{(w-1)(w+1)} = \frac{2}{w^2-1} \\
& \Rightarrow w(A+B) + (A-B) = 2 \\
& \Rightarrow A+B = 0, \quad A-B = 2 \\
& \Rightarrow 2A = 2 \Rightarrow A = 1 \Rightarrow B = -1 \\
& \Rightarrow \frac{2}{w^2-1} = \frac{1}{w-1} - \frac{1}{w+1}
\end{aligned}}$$

$$(*) = \int \left[2 + \frac{1}{w-1} - \frac{1}{w+1} \right] dw = 2w + \log|w-1| - \log|w+1| + C$$

$$= 2w + \log\left|\frac{w-1}{w+1}\right| + C = 2\sqrt{u} + \log\left|\frac{\sqrt{u}-1}{\sqrt{u}+1}\right| + C$$

$$= 2\sqrt{1+e^x} + \log\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C \qquad \square$$

Product rule: $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$

$\hookrightarrow$ multiplication

$$f(x)g(x) = \int (f \cdot g)'(x)\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$\Rightarrow \int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Notation: $\rightsquigarrow \int f\,dg = fg - \int g\,df$

$$\int f(x)\,g'(x)\,dx \quad \begin{bmatrix} f = f(x) & dg = g'(x)\,dx \\ df = f'(x)\,dx & g = g(x) \end{bmatrix}$$

$$= f(x)g(x) - \int f'(x)g(x)\,dx$$

Examples:

$$\int x e^x dx \left[\begin{array}{ll} f = x & dg = e^x dx \\ df = 1 \cdot dx & g = e^x \end{array}\right] = xe^x - \int e^x dx$$

$$= xe^x - e^x + C \quad \square$$

$\sin'(0) = 1$

$\cos(0) = 1$

$del = 1$

$$\int \log(x) dx \left[\begin{array}{ll} f = \log(x) & dg = 1 \cdot dx \\ df = \frac{dx}{x} & g = x \end{array}\right]$$

$$= x \log(x) - \int \frac{x}{x} dx = x \log(x) - x + C \quad \square$$

$$\underline{\int e^x \sin(x) dx} \left[\begin{array}{ll} f = e^x & dg = \sin(x) dx \\ df = e^x dx & g = -\cos(x) \end{array}\right]$$

$$= -e^x \cos(x) + \int e^x \cos(x) dx \left[\begin{array}{ll} f = e^x & dg = \cos(x) dx \\ df = e^x dx & g = \sin(x) \end{array}\right]$$

$$= -e^x \cos(x) + e^x \sin(x) - \underline{\int e^x \sin(x) dx}$$

$$\Rightarrow 2 \int e^x \sin(x) dx = e^x (\sin(x) - \cos(x))$$

$$\Rightarrow \int e^x \sin(x) dx = \frac{1}{2} e^x (\sin(x) - \cos(x)) + C \quad \square$$

$$\int \arctan(x)\,dx \quad \left[\begin{matrix} f = \arctan(x) & dg = 1 \cdot dx \\ df = \dfrac{dx}{1+x^2} & g = x \end{matrix}\right]$$

$$= x\arctan(x) - \int \frac{x\,dx}{1+x^2} \quad \left[\begin{matrix} \text{let } u = x^2 \\ du = 2x\,dx \end{matrix}\right]$$

$$= x\arctan(x) - \frac{1}{2}\int \frac{du}{u+1}$$

$$= x\arctan(x) - \frac{1}{2}\log|u+1| + C$$

$$= x\arctan(x) - \frac{1}{2}\log|x^2+1| + C \quad \square$$

Ordinary Differential Equations

In general a n-th order ODE is an equation of the form (here $y = f(x)$)

$$F(x, y, y', \dots, y^{(n)}) = 0$$

e.g. $x^2y^2 + 2yy'\sin(x) + e^x = 0$ (1st order ODE)

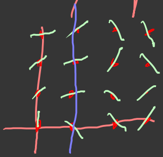
↪ almost certainly there is no general solution that can be cleanly written e.g. $\int \sqrt{\sin(x)}\,dx = Si(x)$

Integration is solving the problem $\boxed{\frac{dy}{dx} = f(x)}$

Generalise this by one step to $\boxed{\frac{dy}{dx} = f(x,y)}$

Geometric significance to this problem:

At every point (a,b) I have the slope $f(a,b)$



A solution to this problem is a curve with the specified tangent direction at every point

This type of problem involves giving direction field or a 1-distribution

e.g. let $q(t)$ denote the quantity of a radioactive substance at time t

so $q'(t) = -\alpha q(t)$ for $\alpha \in (0,1)$

Separable differential equation:

Equations of the form $\frac{dy}{dx} = f(x)g(y)$

eg. $y' = x^2 y^2$ works but $y' = \sin(x+y)$

or even $y' = x+y$

are not of this form

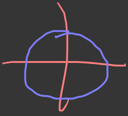
Replace g with $1/g$ so that we have

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$\Rightarrow g(y)\,dy = f(x)\,dx$$

$$\Rightarrow \int g(y)\,dy = \int f(x)\,dx$$

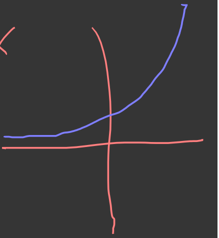
$\Rightarrow$ General solution is $\int f(x)\,dx - \int g(y)\,dy = C$



$x^2+y^2=1$

$$\frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{y} = dx \Rightarrow \int \frac{dy}{y} = \int dx$$



$$\Rightarrow \log|y| = x + C$$

$$\Rightarrow |y| = e^{\log|y|} = e^{x+C} = e^{C}e^{x}$$

$$\Rightarrow y = \pm e^{C}e^{x}$$

More generally $y = Ce^{x}$

Shortcut:

$$\frac{dy}{y} = dx \Rightarrow \int \frac{dy}{y} = \int dx$$

$$\Rightarrow \log(y) = x + \log(C)$$

$$\Rightarrow \quad y = Ce^{x} \quad \square$$

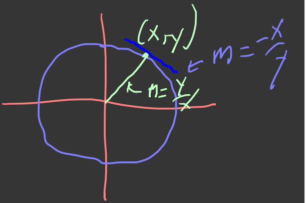
$$\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \log(y) = \log(x) + \log(C)$$

$$= \log(Cx)$$

$$\Rightarrow y = e^{\log(y)} = e^{\log(Cx)} = Cx \quad \square$$

$$\frac{dy}{dx} = \frac{-x}{y}$$



$$\Rightarrow y\,dy = -x\,dx$$

$$\Rightarrow \int y\,dy = -\int x\,dx$$

$$\Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + \frac{1}{2}r^2$$

$$\Rightarrow y^2 + x^2 = r^2$$

$$\frac{dy}{dx} = \frac{(y^2+1)\,x}{y}$$

$$\frac{y\,dy}{dx} = (y^2+1)\,x$$

$$\Rightarrow \frac{y}{y^2+1}\,dy = x\,dx$$

$$\Rightarrow \frac{1}{2}\log(y^2+1) = \frac{1}{2}x^2 + \frac{1}{2}\log(C)$$

$$\Rightarrow y^2+1 = Ce^{x^2}$$

$$\Rightarrow y = \pm\sqrt{Ce^{x^2}-1}$$

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2} \Rightarrow \frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

$$\Rightarrow \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} \Rightarrow \arctan(y) = \arctan(x) + \arctan(C)$$

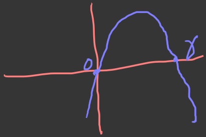
$$\Rightarrow y = \tan(\arctan(y)) = \tan(\arctan(x) + \arctan(C))$$
$$= (*)$$

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

$$(*) = \frac{x + C}{1 - xC}$$

Population dynamics: $r(t)$ represents the number of rabbits in an ecosystem at time t

$$\frac{dr}{dt} = kr(\gamma - r) \quad , \gamma > 0$$



$$\frac{dr}{r(\gamma - r)} = k\,dt \Rightarrow kt = \int \frac{dr}{r(\gamma - r)} = (*)$$

$$\boxed{\begin{aligned} &\frac{A}{r} + \frac{B}{\gamma - r} = \frac{A(\gamma - r) + Br}{r(\gamma - r)} = \frac{r(B-A) + A\gamma}{r(\gamma - r)} \\ &= \frac{1}{r(\gamma - r)} \Rightarrow B - A = 0 \quad A\gamma = 1 \\ &\qquad\qquad \Rightarrow A = 1/\gamma \quad B = 1/\gamma \end{aligned}}$$

$$(*) = \frac{1}{\gamma}\int \left[\frac{1}{r} + \frac{1}{\gamma - r}\right] dr = \frac{1}{\gamma}\left[\log(r) - \log(\gamma - r)\right] + \frac{1}{\gamma}\log(c)$$

$$\Rightarrow k\gamma t = \log\left(c\frac{r}{\gamma - r}\right)$$

$$\Rightarrow c\frac{r}{\gamma - r} = e^{k\gamma t}$$

$$\Rightarrow Cr = \gamma e^{k\gamma t} - re^{k\gamma t}$$

$$\Rightarrow (C + e^{k\gamma t}) r = \gamma e^{k\gamma t}$$

$$\Rightarrow r = \frac{\gamma e^{k\gamma t}}{C + e^{k\gamma t}} = \frac{\gamma}{1 + Ce^{-k\gamma t}} \quad \square$$

Linear ODEs:

A linear ODE of order n is an equation of the form:

$$(*) \quad \boxed{f_n(x) y^{(n)} + \dots + f_1(x) y' + f_0(x) y = g(x)}$$

This is called homogeneous if $g = 0$.

e.g. I can't have y^2 or yy'

so $y^2 e^x + yy' + \sin(x) = 0$ is not linear

but $e^x y'' + \sin(x) y' + 2y = \arctan(x)$ is linear.

Suppose $f_n \cdot y_I^{(n)} + \dots + f_0 \cdot y_I = g(x)$

and $f_n \cdot y_H^{(n)} + \dots + f_0 \cdot y_H = 0$

then $f_n \cdot (y_I^{(n)} + y_H^{(n)}) + \dots + f_0 \cdot (y_I + y_H) = g(x)$

thus I can add any solution of the homogeneous problem to a particular solution of the inhomogeneous problem (and this will still be a solution)

Also, I can add and scale solutions of the homogeneous problem

In particular first order linear ODEs are equations of the form

$$\boxed{f(x)y' + g(x)y = h(x)}$$

Or in equal generality

$$\boxed{y' + f(x)y = g(x)}$$

$$\Rightarrow y' = -f(x)y + g(x)$$

What about the homogeneous case?

i.e. $y' + f(x)y = 0$

$$\Rightarrow \frac{dy}{dx} = -f(x)y$$

$$\Rightarrow \frac{dy}{y} = -f(x)dx$$

$$\Rightarrow \log(y) = -\int f(x)dx + \log(C)$$

$$\Rightarrow y = Ce^{-\int f(x)dx} \quad \square$$