MAT 126 Midterm Solutions

1. (a)

$$\int (x^3+2)^2\,dx = \int \left[x^6+4x^3+4\right]dx = \frac{1}{7}x^7 + x^4 + 4x + C \qquad \square$$

1. (b)

$$\begin{array}{r l}
 & 2x^2 - 3x + 1 \\ \hline
x^2+1 \big) & 2x^4 - 3x^3 + 3x^2 - 3x + 1 \\
 & -2x^4 - 2x^2 \\ \hline
 & -3x^3 + x^2 - 3x \\
 & 3x^3 + 3x \\ \hline
 & x^2 + 1 \\
 & - x^2 - 1 \\ \hline
 & 0
\end{array}$$

$$\int \frac{2x^4-3x^3+3x^2-3x+1}{x^2+1}\,dx = \int \left[2x^2-3x+1\right]dx$$

$$= \frac{2}{3}x^3 - \frac{3}{2}x^2 + x + C \qquad \square$$

1. (c)

$$\int \frac{\cos(x)}{\sec(x)-\tan(x)}\,dx = \int \frac{\cos(x)\,(\sec(x)+\tan(x))}{\sec^2(x)-\tan^2(x)}\,dx$$

$$= \int \frac{\cos(x)\left(\frac{1}{\cos(x)}+\frac{\sin(x)}{\cos(x)}\right)}{1}\,dx = \int [1+\sin(x)]\,dx$$

$$= x - \cos(x) + C \qquad \square$$

2. (a)

$$\int e^{e^x} e^x\,dx \left[\begin{array}{l}\text{Let } u = e^x \\ du = e^x\,dx\end{array}\right] = \int e^u\,du = e^u + C = e^{e^x} + C \qquad \square$$

2. (b)

$$\int \frac{x^2}{\sqrt{1-x^6}}\,dx \left[\begin{array}{l}\text{Let } u = x^3 \\ du = 3x^2\,dx\end{array}\right] = \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{3}\arcsin(u) + C = \frac{1}{3}\arcsin(x^3) + C \qquad \square$$

3. (a)

$$\int x^2 \log(x)\,dx \left[\begin{array}{ll} u = \log(x) & dv = x^2\,dx \\ du = \frac{1}{x}\,dx & v = \frac{1}{3}x^3 \end{array}\right]$$

$$= \frac{1}{3}x^3\log(x) - \frac{1}{3}\int x^2\,dx = \frac{1}{3}x^3\log(x) - \frac{1}{9}x^3 + C \qquad \square$$

3. (b)

$$\int x^2 \sin(x)\, dx \quad \begin{bmatrix} u = x^2 & dv = \sin(x)\, dx \\ du = 2x\, dx & v = -\cos(x) \end{bmatrix}$$

$$= -x^2\cos(x) + 2\int x\cos(x)\, dx \quad \begin{bmatrix} u = x & dv = \cos(x)\, dx \\ du = dx & v = \sin(x) \end{bmatrix}$$

$$= -x^2\cos(x) + 2x\sin(x) - 2\int \sin(x)\, dx$$

$$= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$$

□

4. (a)

$$\frac{x+1}{x^3+x^2-6x} = \frac{x+1}{x(x-2)(x+3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}$$

$$= \frac{A(x-2)(x+3) + Bx(x+3) + Cx(x-2)}{x(x-2)(x+3)}$$

$$= \frac{A(x^2+x-6) + B(x^2+3x) + C(x^2-2x)}{x(x-2)(x+3)}$$

$$= \frac{(A+B+C)x^2 + (A+3B-2C)x - 6A}{x(x-2)(x+3)}$$

$$\Rightarrow A+B+C = 0, \quad A+3B-2C = 1, \quad -6A = 1$$

$$\Rightarrow A = \frac{-1}{6}, \quad B = \frac{3}{10}, \quad C = \frac{-2}{15}$$

$$\int \frac{x+1}{x^3+x^2-6x}\, dx = -\frac{1}{6}\int \frac{1}{x}dx + \frac{3}{10}\int \frac{1}{x-2}dx - \frac{2}{15}\int \frac{1}{x+3}dx$$

$$= -\frac{1}{6}\log|x| + \frac{3}{10}\log|x-2| - \frac{2}{15}\log|x+3| + C$$

□

4. (b)

$$\frac{x^3+4x^2+10}{(x-1)^2(x^2+4)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4}$$

$$= \frac{A(x-1)(x^2+4) + B(x^2+4) + (Cx+D)(x-1)^2}{(x-1)^2(x^2+4)}$$

$$= \frac{A(x^3-x^2+4x-4) + B(x^2+4) + (Cx+D)(x^2-2x+1)}{(x-1)^2(x^2+4)}$$

$$= \frac{(A+C)x^3 + (-A+B-2C+D)x^2 + (4A+C-2D)x - 4A+4B+D}{(x-1)^2(x^2+4)}$$

$$\Rightarrow A+C = 1, \quad -A+B-2C+D = 4,$$

$$4A+C-2D = 0, \quad -4A+4B+D = 10$$

$$\Rightarrow A = 1, \quad B = 3, \quad C = 0, \quad D = 2$$

$$\int \frac{1}{x^2+4}\,dx \left[\begin{array}{c}\text{Let } x = 2\tan(u)\\ dx = 2\sec^2(u)\,du\end{array}\right] = \int \frac{2\sec^2(u)}{4(1+\tan^2(u))}\,du = \frac{1}{2}\int du$$

$$= \frac{1}{2}u + C = \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

$$\int \frac{x^3+4x^2+10}{(x-1)^2(x^2+4)}\,dx = \int \frac{1}{x-1}\,dx + 3\int \frac{1}{(x-1)^2}\,dx + 2\int \frac{1}{x^2+4}\,dx$$

$$= \log|x-1| - \frac{3}{x-1} + \arctan\left(\frac{x}{2}\right) + C \qquad \square$$

5. (a)

$$\int \sin^4(x)\,dx = \int \left(\sin^2(x)\right)^2\,dx = \int \left(\frac{1}{2}\left[1-\cos(2x)\right]\right)^2\,dx$$

$$= \frac{1}{4}\int \left[1 - 2\cos(2x) + \cos^2(2x)\right]\,dx$$

$$= \frac{1}{4}\int \left[1 - 2\cos(2x) + \frac{1}{2}\left(1+\cos(4x)\right)\right]\,dx$$

$$= \int \left[\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right]\,dx$$

$$= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$$

5. (b)

$$\int \frac{\sqrt{9+x^2}}{x^2}\,dx \left[\begin{array}{c}\text{Let } x = 3\tan(u)\\ dx = 3\sec^2(u)\,du\end{array}\right] = \int \frac{9\sqrt{1+\tan^2(u)}\sec^2(u)}{9\tan^2(u)}\,du$$

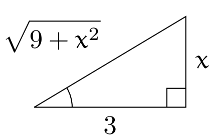
$$= \int \frac{\sec^3(u)}{\tan^2(u)}\,du = \int \frac{1}{\cos^3(x)}\frac{\cos^2(u)}{\sin^2(u)}\,du = \int \sec(u)\csc^2(u)\,du$$

$$= \int \sec(u)(1+\cot^2(u))\,du = \int \left[\sec(u) + \frac{1}{\cos(u)}\frac{\cos^2(u)}{\sin^2(u)}\right]\,du$$

$$= \int \sec(u)\,du + \int \csc(u)\cot(u)\,du$$

$$= \log|\sec(u)+\tan(u)| - \csc(u) + C$$

$$= \log|\sec(\arctan(x/3)) + x/3| - \csc(\arctan(x/3)) + C$$



$$= \log\left|\frac{\sqrt{9+x^2}+x}{3}\right| - \frac{\sqrt{9+x^2}}{x} + C \qquad \square$$

5. (c)

$$\int \log\left(x+\sqrt{x^2-1}\right)\,dx \quad \left[\begin{array}{c}\text{Let } x=\sec(u)\\ dx=\sec(u)\tan(u)\,du\end{array}\right]$$

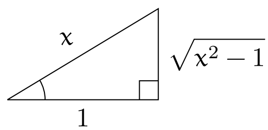
$$=\int \log\left(\sec(u)+\sqrt{1-\sec^2(u)}\right)\sec(u)\tan(u)\,du$$

$$=\int \log\left(\sec(u)+\tan(u)\right)\sec(u)\tan(u)\,du$$

$$\left[\begin{array}{cc}u=\log\left(\sec(u)+\tan(u)\right) & dv=\sec(u)\tan(u)\,du\\ du=\sec(u)\,du & v=\sec(u)\end{array}\right]$$

$$=\sec(u)\log\left(\sec(u)+\tan(u)\right)-\int \sec^2(u)\,du$$

$$=\sec(u)\log\left(\sec(u)+\tan(u)\right)-\tan(u)+C$$



$$=x\log\left(x+\sqrt{x^2-1}\right)-\sqrt{x^2-1}+C \qquad \square$$

5. (d)

$$\int \frac{1}{3+2\cos(x)+\sin(x)}\,dx \quad \left[\text{Let } u=\tan(x/2)\right]$$

$$=\int \frac{1}{3+2\frac{1-u^2}{1+u^2}+\frac{2u}{1+u^2}}\,\frac{2}{1+u^2}\,du$$

$$=\int \frac{2}{3(1+u^2)+2(1-u^2)+2u}\,du$$

$$=\int \frac{2}{u^2+2u+5}\,du=\int \frac{2}{(u+1)^2+4}\,du$$

$$=\frac{1}{2}\int \frac{1}{\left(\frac{u+1}{2}\right)^2+1}\,du \quad \left[\begin{array}{c}\text{Let } w=(u+1)/2\\ dw=1/2\,du\end{array}\right]$$

$$=\int \frac{1}{w^2+1}\,dw=\arctan(w)+C$$

$$=\arctan\left(\frac{u+1}{2}\right)+C=\arctan\left(\frac{\tan(x/2)+1}{2}\right)+C \qquad \square$$

6. (a)

$$F(x)=\int_{-x}^{x^2}\log(\cos(t))\,dt=\int_0^{x^2}\log(\cos(t))\,dt-\int_0^{-x}\log(\cos(t))\,dt$$

$$F'(x)=2x\log(\cos(x^2))+\log(\cos(-x)) \qquad \square$$

6. (b)

$$F(x) = \int_0^{\int_2^{x^2} \frac{1}{1+t^2}\,dt} t^5\,dt$$

$$F'(x) = \left(\int_2^{x^2} \frac{1}{1+t^2}\,dt\right)^5 \frac{2x}{1+x^4} \qquad \square$$

7. Since f is increasing on the interval $[0,1]$, we have, for any $X = [x,y] \subseteq [0,1]$, that $\inf_X f = f(x)$ and $\sup_X f = f(y)$. Now, let P_n be the n-th uniform partition, such that $P_n \equiv \left(0 < \frac{1}{n} < \ldots < \frac{n-1}{n} < \frac{n}{n}\right)$. We then have:

$$\begin{aligned}
U(f, P_n) &= \sum_{k=1}^{n} \left(\frac{k}{n} - \frac{k-1}{n}\right) \sup_{[\frac{k-1}{n}, \frac{k}{n}]} f \\
&= \sum_{k=1}^{n} \frac{1}{n}\left(\frac{k}{n}\right) \\
&= \frac{1}{n^2}\left(\sum_{k=1}^{n} k\right) \\
&= \frac{1}{n^2}\frac{n(n+1)}{2} \\
&= \frac{1}{2}\left(1 + \frac{1}{n}\right)
\end{aligned}$$

Using the summation formula given in the notes. We similarly have:

$$\begin{aligned}
L(f, P_n) &= \sum_{k=1}^{n} \left(\frac{k}{n} - \frac{k-1}{n}\right) \inf_{[\frac{k-1}{n}, \frac{k}{n}]} f \\
&= \sum_{k=1}^{n} \frac{1}{n}\left(\frac{k-1}{n}\right) \\
&= \frac{1}{n^2}\left(\sum_{k=1}^{n-1} k\right) \\
&= \frac{1}{n^2}\frac{(n-1)\,n}{2} \\
&= \frac{1}{2}\left(1 - \frac{1}{n}\right)
\end{aligned}$$

The limits $\lim_n L(f, P_n)$ and $\lim_n U(f, P_n)$ may now be calculated using limit laws. All that we need to know is that $\lim_n 1/n = 0$ and the rest will follow via several applications of the laws.

$$
\begin{aligned}
\lim_n L(f, P_n) &= \lim_n \frac{1}{2}\left(1 - \frac{1}{n}\right) \\
&= \frac{1}{2}\left(1 - \lim_n \frac{1}{n}\right) \\
&= \frac{1}{2} \lim_n (1 - 0) = \frac{1}{2} \\
\lim_n U(f, P_n) &= \lim_n \frac{1}{2}\left(1 + \frac{1}{n}\right) \\
&= \frac{1}{2}\left(1 + \lim_n \frac{1}{n}\right) \\
&= \frac{1}{2} \lim_n (1 + 0) = \frac{1}{2}
\end{aligned}
$$

Hence $\lim_n L(f, P_n) = \lim_n U(f, P_n) = \frac{1}{2}$, so $\int_0^1 f(x)\,dx = \frac{1}{2}$. □

(Note that I copied this near-verbatim from the notes; you've already seen this solution twice.)